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Supersimetría exótica basada en sistemas no lineales integrables.

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SUPERSIMETRÍA EXÓTICA BASADA EN SISTEMAS

NO LINEALES INTEGRABLES

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Resumen

En la base de la construcción de mecánica cuántica supersimétrica en 1+1D está el modelo de juego de Witten de mecánica pseudo-clásica. La versión cuantizada de este modelo es análoga a la definición de la transformación de Darboux entre operadores Hamiltonianos de tipo Schrödinger, la cual corresponde a una transformación espectral covariantes entre potenciales y autoestados de pares de este tipo de operador diferencial. Una generalización de esta tranformación, llamada transformación de Crum-Darboux, permite definir un tipo peculiar de supersimetrías, las cuales presentan coeficientes de estructura dependientes en forma no lineal en el Hamiltoniano. Esta tesis está dedicada a la investigación de una estructura supersimétrica exótica, compuesta por pares de operadores de tipo Schrödinger en 1+1D con supersimetría nolineal y extendida debido a la existencia de una integral bosónica adicional llamada integral de Lax-Novikov, la cual juega un rol importante en teoría de sistemas integrables y relaciona los pares de potenciales de dichos operadores de Schrödinger con supersimetría extendida con los potenciales transparentes y soluciones multisolitónicas en fondo finite-gap de la jerarquía inhomogénea de ecuaciones estacionarias de Korteweg de Vries (s-KdV). Los efectos de disperción entre solitones son responsables de deformaciones espontáneas del superálgebra e incluso rompimientos espontáneos completos o parciales de supersimetría.

Las características de las simetrías de este tipo de sistemas los convierten en exactamente solubles, las propiedades algebráicas de los autoestados de dichos Hamiltonianos permiten construir soluciones analíticas a sistemas de ecuaciones para bosones con acople no lineal, mientras que para superálgebras con integrales fermiónicas de primer orden, los autoestados que diagonalizan dichas integrales fermiónicas solucionan sistemas de ecuaciones de interacción de fermiones con acople no lineal, es posible demostrar que el correspondiente superpotencial es una solución de la jerarquía de ecuaciones inhomogéneas y estacionarias de Korteweg de Vries modificada (s-mKdV). Este resultado obtiene aplicaciones importantes en la búsqueda de soluciones para el modelo de juego de QCD en 1+1D introducido por Gross y Neveu, el cual corresponde a un modelo de interacción no lineal entre fermiones con rompimiento dinámico de simetría.

Palabras claves: supersimetría extendida, exactamente soluble, interacción no lineal, Sistemas integrables no lineales, modelo de Gross-Neveu.

Abstract

In the basis of the construction of supersymmetric quantum mechanics in 1+1D is the Witten's toy model of pseudo-classical mechanics. The quantized version of this model is analogous to the definition of Darboux transformation which corresponds to a covariant spectral transformation between the potentials and eigenstates of Hamiltonian operators. A generalization of this transformation is the called Crum-Daboux transformation, this allows to define a particular group of supersymmetries which present structure coefficients depend on the non-linear form in the Hamiltonian. This thesis is dedicated to research an exotic supersymmetric structure, composed by pairs of Schrodinger operators in 1+1D with nonlinear supersymmetry extended because of the existence of an additional bosonic integral called the Lax-Novikov integral which plays an important role in the integrable systems theory and relates the pair of potentials of such Schrodinger operators with extended supersymmetry with transparent potentials and multisolitonic solutions in finite-gap background of the inhomogeneous hierarchy of the stationary Korteweg-de Vries equations (s-KdV). The effects of dispersion between solitons are responsible of spontaneous deformation of the superalgebra, as well as of the partial or complete supersymmetry breaking.

The characteristics of the symmetries of this kind of systems make them in exactly solvable; the algebraic properties of the eigenstates of such Hamiltonians allow construct analytic solutions to systems of equations for bosons with nonlinear interaction, whereas for superalgebras with fermionic integrals of first order, the eigenstates that diagonalized these fermionic integrals solve systems of equations for nonlinear interaction between coupled fermions; can be demonstrated the corresponding superpotential are solutions of the inhomogeneous stationary hierarchy of modified Korteweg-de Vries equations (s-mKdV). This result gets applications in the search of solutions for the toy model of QCD in 1+1D introduced by Gross and Neveu, which corresponds to a model of non-linear interaction among fermions with dynamic symmetry breaking.

Keywords: extended supersymmetry, exactly solvable, non-linear interaction, non-linear integrable systems, Gross-Neveu model.

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Introducción

Solitones y objetos relacionados topológicamente no triviales, tales como kinks, instantones, vórtices, monopolos y paredes de dominio, juegan un papel importante en diversas áreas de la física, la ingeniería y la biología [50, 46, 45]. Las transformaciones de Darboux y Backlund, con su origen en la teoría de problemas lineales de Sturm-Liouville y la geometría diferencial clásica, ha demostrado ser muy efectivas en su estudio [53, 14]. La transformación de Darboux [53], por otra parte, se encuentra en la base de la construcción de mecánica cuántica supersimétrica [20, 21].

Los solitones aparecen como soluciones de las ecuaciones de campos clásicas no lineales integrables, y por medio de la representación de par de Lax [47] , están relacionados con sistemas cuánticos completamente transparentes con un número finito de bandas [18, 26]. Estas familias de sistemas cuánticos se caracterizan por integrales de movimiento de altos ordenes en derivadas, y como mostraremos a lo largo de esta tesis, extensiones supersimétricas de ellos poseen algunas propiedades peculiares. Uno de los resultados más exóticos es el surgimiento de simetrías con coeficientes de estructura no lineales en los elementos centrales del álgebra de integrales de movimiento, conocidas como supersimetrías no lineales [33, 30, 17, 16, 23, 43]. Estructuras supersimétricas exóticas de sistemas transparentes y con número finito de bandas han encontrado recientemente algunas aplicaciones físicas interesantes que se encuentran en [25, 22, 41, 56, 55].

El ejemplo más conocido de los sistemas transparentes está dada por una jerarquía llamada potenciales Pöschl-Teller. El Hamiltoniano de Schrödinger con potencial de Pöschl-Teller $V_n = -n(n+1)k^2\mathrm{sech}^2 k(x-x_0)$ corresponde a una familia particular de Hamiltonianos transparentes que posee, n estados ligados. Una de sus aplicasiones recae en el control de la estabilidad de las soluciones tipo kinks en sine-Gordon, φ^4 y otros modelos exóticos de teoría de campos (1 + 1) dimensionales [50, 45, 49, 51, 38, 44, 13, 3]. Estos sistemas también aparecen en el modelo de Gross-Neveu [48, 36] y corresponde a soluciones de la jerarquia de ecuaciones estacionarias de Korteweg-de Vries (s-KdV) [46, 53, 10].

En la presente tesis se estudia la supersimetría exótica que describen algunos pares de sistemas transparentes con un número finito de solitones. Se investiga una clase peculiar de sistemas supersimétricos, mecánico cuánticos unidimensionales, descritos por una matriz Hamiltoniana

2×2

$$\mathcal{H} = \begin{pmatrix} -\frac{d^2}{dx^2} + V_+(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + V_-(x) \end{pmatrix}, \tag{I}$$

con

$$V_+(x) \quad \text{y} \quad V_-(x), \tag{II}$$

soluciones multi-solitónicas de las ecuaciones de la jerarquía s-KdV, cada uno en función de sus propios conjuntos de parámetros espectrales. Una de las posibles interpretaciones físicas del sistema (I), (II) es ser considerado como el Hamiltoniano para una partícula de espín-$1/2$ no-relativista, con potencial dependiente del espín de una forma especial (que no introduce qiro de espín).

Por otro lado, en el cuadro de Heisenberg de mecánica cuántica en el cual la dependencia del tiempo recae en los operadores y no en los estados, la evolución del operador $\hat{I}$, cuantización de alguna función en el espacio de fase I (momento, energía, posición, etc), está regida por la ecuación de movimiento de Heisenberg

$$\frac{d\hat{I}}{dt} = \frac{i}{\hbar}[\hat{I}, H] + \frac{\partial \hat{I}}{\partial t}, \tag{III}$$

con $[,]$ el conmutador. De mecánica clásica y el célebre teorema de Noether, nos enseña que las cantidades conservadas o integrales de movimiento están asociadas con las simetrías del sistema. En mecánica cuántica para sistemas estacionarios, la condición de integral $\frac{d\hat{I}}{dt} = 0$ simplemente se reduce a $[\hat{I}, H] = 0$.

En el proceso de cuantización de fermiones es necesario una formulación de mecánica clásica con variables anticonmutantes o variables de Grassman, conocida como mecánica pseudo-clásica. En este formalismo se deben tratar vínculos entre las coordenadas del espacio de fase, debiendo reemplazar los corchetes de Poisson utilizados en la mecánica Hamiltoniana por los corchetes de Dirac, técnica utilizada para cuantizar sistemas degenerados. En este caso el proceso usual de cuantización distingue entre dos tipos de integrales: integrales bosónicas $\hat{I}_B$ e integrales fermiónicas $\hat{I}_F$. Además el proceso provee de un operador Γ de graduación $\mathbb{Z}_2$, $\Gamma^2 = 1$ el cual distingue dichas integrales en la forma

$$[\hat{I}_B, \Gamma] = 0, \quad \{\hat{I}_F, \Gamma\} = 0. \tag{IV}$$

O sea, el operador de graduación conmuta con las integrales bosónicas y anticonmuta con las integrales fermiónicas. Además también fija las reglas del superálgebra

$$[\hat{I}_B, \hat{I}'_B] \sim \hat{I}''_B, \quad [\hat{I}_B, \hat{I}_F] \sim \hat{I}'_F, \tag{V}$$

$$\{\hat{I}_F, \hat{I}'_F\} \sim \hat{I}_B, \tag{VI}$$

Un sistema (I) de forma general, con potenciales $V_+(x)$ y $V_-(x)$ elegidos de forma arbitraria, tiene sólo una integral bosónica en forma de la matriz diagonal de Pauli σ_3, $[H, \sigma_3] = 0$.

Para una elección especial de los potenciales $V_\pm = W^2(x) \pm \frac{dW}{dx}$, esta simetría trivial se extiende a una estructura supersimétrica relacionada con las integrales de movimiento no triviales adicionales $Q_1 = -i\frac{d}{dx}\sigma_1 + \sigma_2 W(x)$, $Q_2 = i\sigma_3 Q_1$. Las cuales generan una estructura superalgebraica de Lie lineal en $\mathcal{H}$, $\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}$, $[\mathcal{H}, Q_a] = 0$, $a, b = 1, 2$, con la integral σ_3 jugando el papel de operador de graduación $\mathbb{Z}_2$, conmutando con las integrales bosónicas y anticonmutando con las integrales fermiónicas, $[\sigma_3, \mathcal{H}] = 0$, $\{\sigma_3, Q_a\} = 0$.

Esta estructura superalgebraica lineal aparece, en partícular, en el problema de Landau para el electrón no relativista, donde el superpotencial es una función lineal $W(x) = \omega x$, y (I) es el hamiltoniano de un superoscilador, ver [21]. La existencia de la estructura supersimétrica lineal es equivalente a la condición de que las componentes superior e inferior de la matriz Hamiltoniana, $H_\pm = -\frac{d^2}{dx^2} + V_\pm$, esten relacionados por operadores de entrelazamiento de Darboux (generadores de la transformación de Darboux), o sea $H_+A_+ = A_+H_-$, $H_-A_- = A_-H_+$, siendo los operadores de entrelazamiento de Darboux operadores de primer orden definidos en la forma $A_+ = \frac{d}{dx} + W(x)$ y $A_- = A_+^\dagger = -\frac{d}{dx} + W(x)$. Con esta observación, la construcción se puede generalizar a supersimetrías no lineales [1, 2] si los potenciales V_+ y V_- son tales que los hamiltonianos correspondiente están entrelazandos por relaciones de la misma forma, pero con A_+ y $A_- = A_+^\dagger$ operadores diferenciales de orden $\ell > 1$(transformación de Crum-Darboux). Si esto ocurre, el sistema $\mathcal{H}$ posee las siguientes supercargas nilpotentes $Q_+ = A_+\sigma_+ = \frac{1}{2}(Q_2 + iQ_1)y$ $Q_- = A_-\sigma_- = Q_+^\dagger$, $[Q_\pm H] = 0$, $Q_\pm^2 = 0$, donde $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. Supercargas que generan una supersimetría no lineal de la forma $\{Q_a, Q_b\} = 2\delta_{ab}P_\ell(\mathcal{H})$, donde $P_\ell(\mathcal{H})$ es un polinomio de orden ℓ.

El ejemplo más simple de un sistema con supersimetría no lineal es proporcionado por una generalización del superoscilador $\mathcal{H} = b^+b^- + \ell\frac{1}{2}(1 + \sigma_3)$, para el cual $A_+ = (b^-)^\ell$, $b^\pm$ son los operadores de creación-aniquilación habituales del oscilador bosónico, y el orden del polinomio $P_\ell(\mathcal{H}) = \prod_{j=0}^{\ell-1}(H - j\omega)$ es ℓ , véase ref. [40].

La peculiaridad de los sistemas (I), (II) que se estudian en esta tesis es que los potenciales multi-solitónicos (II) transparentes poseen una integral de Lax-Novikov análoga a la integral de momento lineal $P = -i\frac{d}{dx}$ de la partícula libre. Por una conocida construcción basada en las transformaciones de Crum-Darboux, tales potenciales y sus correspondientes integrales se pueden obtener a partir de soluciones álgebro-geométricas de la jerarquía de ecuaciones estacionarias de Korteweg-de Vries (s-KdV) y su formulación de par de Lax. Se verá que, como consecuencia, el sistema transparente multi-solitónico extendido (I) en algunos casos puede ser descrito por una estructura supersimétrica exótica dependiente de los datos de dispersión que incluye no sólo uno, sino dos pares de supercargas fermiónicas matriciales, anti-diagonales, y dos integrales bosónicas diagonales, relacionadas con la existencia de las integrales de Lax-Novikov de orden impar. Las supercargas en caso general son operadores diferenciales matriciales de

orden superior, dos de los cuales son del orden par, y otras dos supercargas son de orden impar. El superálgebra generada por las cuatro supercargas es no lineal, e incluye en su estructura las integrales de movimiento bosónicas adicionales.

Demostraremos una peculiaridad de tales estructuras supersimétricas de sistemas multi-solitónicos extendidos y ésta es que experimenta cambios radicales en dependencia de las relaciones entre los dos conjuntos de parámetros de dispersión del par de potenciales II: el orden diferencial de las supercargas puede cambiar, y en el caso por completo isospectral una de las integrales bosónicas adicionales puede transformarce en una carga central de la correspondiente superálgebra no lineal.

Analizando las distintas facetas de dicha supersimetría nos encontramos con una familia especial para la cual un par de supercargas se reduce a operadores diferenciales matriciales de primer orden. Estas supercargas de primer orden y $\mathcal{H}$ forman entre sí un superálgebra lineal. En tal caso, las supercargas de primer orden pueden ser reinterpretadas como hamiltonianos de una partícula de Dirac con potencial transparente con número finito de bandas y estados ligados, cuyo espectro es completamente simétrico. Una reinterpretación nos proporciona soluciones de multi-kink-anti-kink en fondo cuasi-periódico, para el modelo de Gross-Neveu por medio del sistema de Bogoliubov-de Gennes. Linealizado el superpotencial toma rol de condensado cuadrático de fermiones. También puede ser interpretado como parámetro de orden en poliacetileno, o una función de distancia en estructuras cristalinas como el grafeno. En este caso la existencia de una integral que es un elemento central del superálgebra del sistema (I), corresponde a la existencia de una integral de movimiento para el Hamiltoniano de Dirac o de Bogoliubov de Gennes, dependiendo del sistema físico de interés, la formulación de par de Lax indica que dicho pontencial escalar de Dirac corresponde a una solución de la jerarquía de ecuaciones de Korteweg-de Vries modificadas.

El modelo de mecánica cuántica supersimétrica de Witten esta representado por el hamiltoniano

$$\begin{aligned}
\mathcal{H}_W &= \begin{pmatrix} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x) + E_1 & 0 \\ 0 & -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W(x)^2 + \frac{\hbar}{\sqrt{2m}}W'(x) + E_1 \end{pmatrix} \\
&= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W(x)^2 + E_1 - \sigma_3\frac{\hbar}{\sqrt{2m}}W'(x) \\
&= \begin{pmatrix} H_0 & 0 \\ 0 & H_1\,, \end{pmatrix}\,, \qquad \text{(VII)}
\end{aligned}$$

el cual posee dos integrales fermiónicas

$$\mathcal{S}_{W,1} = \begin{pmatrix} 0 & -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\ \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) & 0 \end{pmatrix} \tag{VIII}$$

$$= \begin{pmatrix} 0 & A_1^\dagger \\ A_1 & 0 \end{pmatrix}, \quad \mathcal{S}_{W,2} = i\sigma_3\mathcal{S}_{W,1}, \tag{IX}$$

según el operador de graduación $\Gamma = \sigma_3$. Acá $W(x)$ es alguna función en $x \in \mathbb{R}$ y hemos definido $W'(x)$ como la derivada de $W(x)$ en x y los operadores de Schrödinger $H_{0,1} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{0,1}(x)$, con $V_{\frac{1\pm1}{2}}(x) = W(x)^2 \pm \frac{\hbar}{\sqrt{2m}}W'(x) + E_1$.

El superálgebra de Lie de estas integrales de movimiento toma la forma

$$[\mathcal{H}_w, \mathcal{S}_{W,a}] = 0, \quad \{\mathcal{S}_{W,a}, \mathcal{S}_{W,b}\} = 2\delta_{ab}(\mathcal{H}_W - E_1), \tag{X}$$

con $a, b = 1, 2$.

No cualquier par de potenciales V_0 y V_1 permiten esta construcción, de hecho ambos deben estar relacionados en la forma $V_1(x) = V_0(x) + \hbar\sqrt{\frac{2}{m}}W'(x)$, que más adelante conoceremos como transformación de Darboux. Si fijamos, por ejemplo, $V_0(x)$ luego $W(x)$ queda definido por la ecuación de Riccati

$$V_0(x) = W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x) + E_1.$$

Además es posible observar que $\Psi_1(0,x) = \exp\left(-\frac{\sqrt{2m}}{\hbar}\int_{x_0}^{x} \mathrm{d}x' W(x')\right)$, es un estado (no necesariamente físico) de H_0 con energía E_1 o que es lo mismo

$$H_0\Psi_1(0,x) = E_1\Psi_1(0,x). \tag{XI}$$

Así podemos definir la transformación de Darboux de un potencial $V_0(x)$ como

$$V_0(x) \to V_1(x) = V_0(x) - \frac{\hbar^2}{m}\left(\ln(\Psi_1(0,x))\right)'', \tag{XII}$$

en el contexto de dichas transformaciones, las componentes de las integrales fermiónicas $\mathcal{S}_W$, $A_1 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$ y su hermítica conjugada $A_1^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$, son conocidos como operadores de entrelazamiento entre H_0 y H_1, ya que debido a las factorizaciones $H_0 - E_1 = A_1^\dagger A_1$ y $H_1 - E_1 = A_1 A_1^\dagger$ se cumplen las relaciones de entrelazamiento

$$A_1 H_0 = H_1 A_1, \quad A_1^\dagger H_1 = H_0 A_1^\dagger. \tag{XIII}$$

Estas identidades juegan un rol fundamental en la resolución del problema espectral de potenciales de gran complejidad, ya que nos permite obtener el espectro de H_1 a partir del de H_0 debido a que si $\Psi(0,x,E)$ es un estado de H_0 con energía E, entonces las identidades de entrelazamiento se traducen en que $\Psi(1,x,E) = A_1\Psi(0,x,E)$ es un estado de H_1 con energía E,

mientras que $A_1^\dagger$ realiza el mapeo inverso. Tendremos problemas en el mapeo del estado $\Psi_1(0,x)$ debido a que A_1 por definición puede ser escrito en la forma

$$A_1 = \Psi_1(0,x)\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\frac{1}{\Psi_1(0,x)}, \tag{XIV}$$

por lo que aniquila el estado $\Psi_1(0,x)$, $(\frac{d}{dx}1 = 0)$, en el espacio de autoestados de H_0, mientras que $A_1^\dagger$ aniquila un estado de misma energía en el espacio de autoestados de H_1. De hecho por definición tenemos que

$$A_1^\dagger = -\frac{1}{\Psi_1(0,x)}\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\Psi_1(0,x), \tag{XV}$$

luego podemos ver que el elemento en el espacio de autoestados de H_1 aniquilado por $A_1^\dagger$ es el estado $\frac{1}{\Psi_1(0,x)}$. En este punto es interesante notar que si $\Psi_1(0,x)$ es un estado sin ceros en x y crece (decrece) exponencialmente hacia ambos infinitos luego $\frac{1}{\Phi_1(0,x)}$ es un estado ligado de H_1, en este caso el potencial $V_1(x)$ tiene un defecto solitónico que soporta dicho estado ligado. Un ejemplo de dicho potencial solitónico está dado por el par partícula libre $H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ y Pöschl-Teller (PT) $H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{m}\kappa^2\mathrm{sech}\left(\kappa^2(x-x_0)\right)$, construido a partir del estado $\Psi(0,x,-\frac{\hbar^2}{2m}\kappa^2) = \cosh(\kappa(x-x_0))$, el cual impica un estado ligado para PT $\Psi(1,x,-\frac{\hbar^2}{2m}\kappa^2) = \mathrm{sech}(\kappa(x-x_0))$

Este ejemplo es bastante especial debido a la existencia de una integral adicional para la partícula libre dado que el momento lineal $p = -i\hbar\frac{d}{dx}$ cumple el rol de integral bosónica para $H_0 = \frac{p^2}{2m}$ (integral de Lax-Novikov).

El método de vestimiento de Darboux nos permite construir una integral para PT de la forma $P_1 = A_1 p A_1^\dagger$ la cual por definición conmuta con H_1. Este resultado es trivial utilizando las relaciones de entrelazamiento. Es interesante y necesario notar que la ecuación $[H_1, P_1] = 0$ por medio de la representación de par de Lax corresponde a una representacion de la ecuación estacionaria de Korteweg-de Vries y que la identidad $P_1^2 = H_1(H_1 - E_1)^2$ es conocida como la relación de Burchnal-Chaundy entre H_1 y P_1; y provee de un aproach álgebro-geométrico para la búsqueda de soluciones de KdV y, por consiguiente, de potenciales con supersimetría extendida no lineal, dado que nos fija una curva $y^2 = z(z-E_1)^2$ entre los autovalores y de P_1 y z de H_1. En general, es posible encontrar operadores de tipo P de orden mayor, siendo la forma de esta curva en general híper-elíptica degenerada, con ceros de orden uno para bordes de banda y de orden dos para estados ligados (ya que ambos bordes de banda estan en el mismo punto del espectro).

La construcción de la integral P_1 nos permite definir una integral bosónica de orden tres para $\mathcal{H}_W$

$$\mathcal{P}_1 = \begin{pmatrix} (H_0 - E_1)p & 0 \\ 0 & A_1 p A_1^\dagger \end{pmatrix}, \quad \mathcal{P}_2 = \sigma_3\mathcal{P}_1, \tag{XVI}$$

tal que $\mathcal{P}_1$ conmuta con $\mathcal{H}_W$ y $\mathcal{S}_{W,a}$, mientras que el operador $\mathcal{P}_1$ conmuta con $\mathcal{H}_W$ pero su conmutador con los operadores $\mathcal{S}_{W,a}$ define nuevas integrales fermiónicas relacionadas con los operadores de entrelazamiento $Y_1 = A_1 p$ y $Y_1^\dagger = pA_1^\dagger$, construidos a partir de la libertad de H_0 de conmutar con p.

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & Y_1^\dagger \\ Y_1 & 0 \end{pmatrix} \quad \mathcal{Q}_2 = i\sigma_3\mathcal{Q}_1. \tag{XVII}$$

Acá aparece una de las motivaciones principales de esta tesis, debido a que el superálgebra de Lie de estas integrales de movimiento presentan coeficientes de estructura no lineales en el Hamiltoniao del sistema, un ejemplo de esto es la subálgebra de $\mathcal{H}_W$ y $\mathcal{Q}_a$

$$[\mathcal{H}_W, \mathcal{Q}_a] = 0, \quad \{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}\mathcal{H}_W(\mathcal{H}_W - E_1). \tag{XVIII}$$

Aunque éste no es el cuadro más rico, existe una generalización de la transformación de Darboux llamada transformación de Crum-Darboux, la cual consta de construir un nuevo hamiltoniano H_n a partir de un operador no singular $\mathbb{A}_n$ que aniquile, ya no solo un estado de H_0, sino que un conjunto de ellos. Este operador de entrelazamiento ($\mathbb{A}_n H_0 = H_n \mathbb{A}_n$) nos permite construir una superálgebra no lineal que contiene, en su caso más simple, a la superálgebra de Lie lineal del modelo de Witten

$$\mathcal{H}_n = \begin{pmatrix} H_0 & 0 \\ 0 & H_n\,, \end{pmatrix}, \tag{XIX}$$

$$\mathcal{S}_1 = \begin{pmatrix} 0 & \mathbb{A}_n^\dagger \\ \mathbb{A}_n & 0 \end{pmatrix}, \quad \mathcal{S}_2 = i\sigma_3\mathcal{S}_1, \tag{XX}$$

$$[\mathcal{H}_n, \mathcal{S}_a] = 0, \quad \{\mathcal{S}_a, \mathcal{S}_b\} = 2\delta_{ab}\prod_{i=1}^{n}(\mathcal{H}_n - E_i), \tag{XXI}$$

acá E_i con $i = 1, \ldots, n$ son las energías de los estados aniquilados por $\mathbb{A}_n$. Ahora si además H_0 posee una integral adicional como en el ejemplo de la partícula libre, esta superálgebra se verá extendida de forma similar al ejemplo anterior.

La tesis que se defiende en las siguientes páginas se resume en las siguientes afirmaciones

- El estudio de una supersimetría exótica entre sistemas de tipo Schrödinger con potenciales finite-gap y sus deformaciones espectrales, han permitido construir familias infinitas de soluciones multi-paramétricas para diversas ecuaciones diferenciales no lineales útiles en diversas áreas de la fśica, como superconductividad, conducción en polímeros, materia condensada, QCD, entre otras diversas áreas de física de solitones.

- Los sistemas tipo Schrödinger transparentes o los sistemas con número finito de bandas permitidas conectados mediante transformaciones de Crum-Darboux presentan una supersimetria no lineal extendida. Los metodos utilizados para defender esta afirmación son la formulación de par de Lax de la jerarquía de equaciones de KdV y la transformación de Darboux. El texto esta preparado para que la demostración de esta afirmación en el Cap.4 se logre luego de demostrar algunas afirmaciones segundarias.

- Es posible reescribir la integral de Lax-Novikov como cadena de transformaciones de Darboux. La explicación de esto está detras del fenómeno de desplazamiento de Darboux. Dicho fenómeno ocurre dentro de un grupo de potenciales que no cambian su forma funcional salvo algunos parámetros, bajo transformaciones de Darboux, estos potenciales son conocidos como invariantes de forma. Un subconjunto de estos potenciales invariantes de forma son iso-espectrales. En especial los potenciales que poseen una integral de Lax-Novikov poseen transformaciones de Darboux que mantienen invariante la forma y el espectro las cuales llamaremos auto transformaciones de Darboux y solo generan desfases en el potencial inicial. Si se hacen los suficientes es posible desfasar en un ”periodo”lo que combierte al operador de entrelazamiento en una integral de movimiento.
- Es posible hacer deformaciones espectrales a los potenciales con integrales de Lax-Novikov agregando estados ligados en cualquier banda prohibida del espectro del potencial inicial asegurando la existencia de una integral de Lax-Novikov para el potencial deformado. La base que sustenta esta afirmación es vestimiento de Darboux de la integral de Lax-Novikov del sistema inicial.

- La supersimetria exótica de estos sistemas depende de los datos de disperción de cada par de potenciales siendo posible el rompimiento de simetría, reducciones de orden de las integrales de movimiento y deformaciones en el superálgebra de Lie. Detras de esta dinámica en el superálgebra está el límite que conecta autoestados que generan solitones con autoestados que generan auto transformaciones de Darboux, y por otro lado un principio de exclusión observable entre solitones de misma energía.

Esta tesis está basada en las siguientes publicaciones en la revista Physical Review D:

- M. S. Plyushchay, A. Arancibia and L.-M. Nieto, “*Exotic supersymmetry of the kink-antikink crystal, and the infinite period limit,*” Phys. Rev. D **83**, 065025 (2011) [arXiv:1012.4529 [hep-th]].

- A. Arancibia and M. S. Plyushchay, “*Extended supersymmetry of the self-isospectral crystalline and soliton chains,*” Phys. Rev. D **85**, 045018 (2012) [arXiv:1111.0600 [hep-th]].

- A. Arancibia, J. M. Guilarte and M. S. Plyushchay, “*Effect of scalings and translations on the supersymmetric quantum mechanical structure of soliton systems,*” Phys. Rev. D **87**, 045009 (2013) [arXiv:1210.3666 [math-ph]].

- A. Arancibia, J. M. Guilarte and M. S. Plyushchay, “*Fermion in a multi-kink-antikink soliton background, and exotic supersymmetry,*” Phys. Rev. D **88**, 085034 (2013) [arXiv:1309.1816 [hep-th]].

- A. Arancibia and M. S. Plyushchay, "*Transmutations of supersymmetry through soliton scattering, and self-consistent condensates,*" Phys. Rev. D **90** 025008 (2014) [arXiv:1401.6709 [hep-th]].
- A. Arancibia, F. Correa, V. Jakubsky, J. M. Guilarte and M. S. Plyushchay, "*Soliton defects in one-gap periodic system and exotic supersymmetry*," Phys. Rev. D **90**, 125041 (2014) [arXiv:1507.07060 [hep-th]].
- A. Arancibia and M. S. Plyushchay, "*Chiral asymmetry in propagation of soliton defects in crystalline backgrounds*," Phys. Rev. D **92** (2015), 105009 [arXiv:1410.3565 [hep-th]].

Capítulo 1

Supersimetría no lineal en mecánica cuántica

1.1. Transformaciones de Crum-Darboux y supersimetría no lineal

El modelo de juego de mecánica cuántica supersimétrica de Witten es análogo a la definición de transformación de Darboux entre operadores de Schrödinger. Esta transformación permite mapear el espacio de Hilbert de un operador de Schrödinger al espacio de Hilbert de una familia especial de operadores de Schrödinger iso-espectrales con una diferencia de un estado ligado, cero o menos uno.

La transformación de Darboux es generalizada por la transformación de Crum-Darboux, la cual corresponde a la aplicación de sucesivas transformaciones de Darboux. Esta construcción induce la formulación de supersimetría no lineal en la mecánica cuántica.

Nuestra consideración se basa en el método de iteradas transformaciones de Darboux (transformaciones Crum-Darboux) [53], al sistema mecánico cuántico $H_0 = -\frac{d^2}{dx^2} + U_0(x)$,

$$H_n = H_0 + U_n(x), \qquad U_n = -2\frac{d^2}{dx^2}\log \mathbb{W}_n \,. \tag{1.1}$$

Acá $\mathbb{W}_n$ es el Wronskiano de n autoestados formales ψ_j de H_0, $H_0\psi_j = E_j\psi_j$, $E_i \neq E_j$

$$\mathbb{W}_n = \mathbb{W}(\psi_1, \ldots, \psi_n) = \det \mathcal{A}, \qquad \mathcal{A}_{ij} = \frac{d^{i-1}}{dx^{i-1}}\psi_j, \qquad i, j = 1, \ldots, n \,. \tag{1.2}$$

Los autoestados $\Psi_0(x; E) \neq \psi_j$ de H_0, $H_0\Psi_0(x; E) = E\Psi_0(x; E)$, son mapeados en las autofunciones $\Psi_n(x; E)$ de H_n, $H_n\Psi_n(x; E) = E\Psi_n(x; E)$, por medio de la fracción de Wronskianos,

$$\Psi_n(x; E) = \mathbb{W}(\psi_1, \ldots, \psi_n, \Psi_0(E))/\mathbb{W}_n \,. \tag{1.3}$$

Coherentemente con (1.1), elegimos $\mathbb{W}_0 = 1$ y y definimos *prepotentiales* Ω_n, $n = 0, 1, \ldots$,

$$\Omega_n = -\frac{d}{dx}\log \mathbb{W}_n \qquad \Rightarrow \qquad \frac{d}{dx}\Omega_n = \frac{1}{2}U_n \,. \tag{1.4}$$

A continuación, introducimos los operadores diferenciales de primer orden

$$A_n = \frac{d}{dx} + \mathcal{W}_n \,, \qquad \mathcal{W}_n = \Omega_n - \Omega_{n-1} \,. \tag{1.5}$$

Estos operadores y sus conjugados factorizan los sistemas H_{n-1} y H_n

$$A_n^\dagger A_n = H_{n-1} - E_n, \qquad A_n A_n^\dagger = H_n - E_n, \tag{1.6}$$

y los entrelazan,

$$A_n H_{n-1} = H_n A_n\,, \qquad A_n^\dagger H_n = H_{n-1} A_n^\dagger\,. \tag{1.7}$$

El operador A_n puede ser representado equivalentemente en la forma

$$A_j = (A_{j-1}\ldots A_1\psi_j)\frac{d}{dx}\frac{1}{(A_{j-1}\ldots A_1\psi_j)} = \frac{d}{dx} - \left(\frac{d}{dx}\ln(A_{j-1}\ldots A_1\psi_j)\right)\,, \tag{1.8}$$

acá $A_1 = \psi_1\frac{d}{dx}\frac{1}{\psi_1}$ y $A_{j-1}\ldots A_1\psi_j$ es un autoestado de autovalor E_j para H_{j-1} cualquier otro (físico o no físico) autoestado $\Psi_{j-1}(E)$ de H_{j-1}, $H_{j-1}\Psi_{k-1}(E) = E\Psi_{n-1}(E)$, es mapeado por A_j en el siguiente autoestado de H_j

$$\Psi_j(E) = A_j\Psi_{j-1}(E)\,, \tag{1.9}$$

con el mismo autovalor, $H_j\Psi_j(E) = E\Psi_j(E)$.

Por iteración de (1.7), H_n está relacionado con H_0 en la forma

$$\mathbb{A}_n H_0 = H_n\mathbb{A}_n\,, \qquad \mathbb{A}_n^\dagger H_n = H_0\mathbb{A}_n^\dagger\,, \tag{1.10}$$

acá $\mathbb{A}_n$ es el operador diferencial de orden n,

$$\mathbb{A}_n \equiv A_n\ldots A_1\,. \tag{1.11}$$

Sí $\mathbb{W}_n \neq 0$ y $U_0(x)$ es nosingular, para todo $x \in \mathbb{R}$, entonces el sistema extendido $\mathcal{H} = \mathrm{diag}(H_0, H_n)$ es descrito por una supersimetría no lineal dependiente en los datos de dispersión de los estados utilizados en las transformación de Crum-Darboux, existen dos supercargas nilpotentes Z_2-impares (anti-diagonales) $Q_+ = \mathbb{A}_n^\dagger\sigma_+ = \frac{1}{2}(Q_2 + iQ_1)$ y $Q_- = \mathbb{A}_n\sigma_- = Q_+^\dagger$, $[Q_\pm, \mathcal{H}] = 0$, $Q_\pm^2 = 0$, acá $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. Éstas generan un superálgebra de Lie no lineal en la forma $\{Q_a, Q_b\} = 2\delta_{ab}\prod_{\ell=1}^n(\mathcal{H} - E_\ell)$. Este modelo presenta una ruptura espontánea de supersimetría, que depende de los datos espectrales de los estados ψ_i debido a que el operador de Schrödinger tiene una degeneración formal dos para cada energía, por lo que la elección de los estados ψ_i no es arbitraria y en general es una conbinación lineal de una base de un par de estados para cada energía. Hay estados que mediante la transformación de Darboux producen: desfases nolineales o solitones con o sin singularidades. Respecto a esto último, algunos pares de ellos pueden producir un desfase o uno o dos solitones, como veremos más adelante. Cuando el rol del estado ψ_i con la energía más baja sólo es crear un solitón, entonces dicho soliton soporta el estado con energia más baja de $\mathcal{H}$, el cual es aniquilado por todos los elementos en el superálgebra, por lo que la supersimetría es exacta, mientras que cuando su rol es generar un desfase entonces la supersimetría se encuentra rota, excepto en el caso en que este estado es el estado físico de mínima energía de H_0, en este caso el superálgebra puede aniquilar uno o dos estados.

1.1.1. Transformaciones de Crum-Darboux a potenciales transparentes, y la ecuación de Schrödinger no lineal

El Hamiltoniano de Schrödinger $H_n = H_0 + U_n(x)$ de un sistema transparente con n estados ligados puede ser obtenido mediante la aplicacón de una transformación de Crum-Darboux, la cual es una composición de n transformasiones de Darboux, al Hamiltoniano de la partícula libre $H_0 = -\frac{d^2}{dx^2}$. Un Hamiltoniano transparente puede ser escrito entonces en la forma

$$U_n(x) = -2\frac{d^2}{dx^2}\ln \mathbb{W}_n(x), \tag{1.12}$$

en terminos del Wronskiano $\mathbb{W}_n(x) = \mathbb{W}(\psi_1, \ldots, \psi_n)$, $\mathbb{W}(f_1, \ldots, f_n) = \det W_{ij}$, $W_{ij} = \frac{d^{i-1}}{dx^{i-1}} f_j$, el cual es construido a partir de los autoestados *no-físicos* del Hamiltoniano de la partícula libre ψ_j, $H_0\psi_j = -\kappa_j^2\psi_j$, exponencialmente divergentes en los infinitos espaciales

$$\psi_j(x; \kappa_j, \tau_j) = \begin{cases} \cosh \kappa_j(x + \tau_j), & j = \text{odd} \\ \sinh \kappa_j(x + \tau_j), & j = \text{even} \end{cases}. \tag{1.13}$$

Los parametros κ_j son llamados parametros de escala y deben cumplir a condición, $0 < \kappa_1 < \kappa_2 < ... < \kappa_{j-1} < \kappa_n$, mientras que los parametros de translación τ_j, $j = 1, \ldots, n$, pueden ser tomados como valores reales arbitrarios. Esta eleccion especifica de los autoestados del Hamiltoniano de la particula libre (1.13) garantiza que el Wronskiano $\mathbb{W}_n(x)$ es una funcón sin nodos que genera un potencial no singular $2n$-paramétrico (1.12) [5], $U_n = U_n(x; \kappa_1, \ldots, \kappa_n, \tau_1, \ldots, \tau_n)$.

De acuerdo a la construccion de Darboux-Crum, los autoestados $\psi[n, \lambda]$ del operador de Schrödinger H_n, $H_n\psi[n, \lambda] = \lambda\psi[n, \lambda]$, son obtenidos a partir de los autoestados de la partícula libre $\psi[0, \lambda]$, $H_0\psi[0, \lambda] = \lambda\psi[0, \lambda]$ en la forma,

$$\psi[n; \lambda] = \frac{\mathbb{W}(\psi_1, \ldots, \psi_n, \psi[0; \lambda])}{\mathbb{W}(\psi_1, \ldots, \psi_n)}\ . \tag{1.14}$$

Autoestados ligadas (no normalizados) $\psi[n, -\kappa_j^2]$, $j = 1, \ldots, n$, son construidos, a partir de los autoestados de la partícula libre

$$\psi[0, -\kappa_j^2](x) \equiv \psi'_j(x; \kappa_j, \tau_j) = \begin{cases} \sinh \kappa_j(x + \tau_j), & j = \text{odd} \\ \cosh \kappa_j(x + \tau_j), & j = \text{even} \end{cases}. \tag{1.15}$$

Estas funciones (1.15) forman un conjunto complementario a (1.13), $H_0\psi'_j = -\kappa_j^2\psi'_j$. Este conjunto (1.15) puede ser relacionado con (1.13) por medio de una diferenciación

$$\psi'_j(x; \kappa_j, \tau_j) = \frac{1}{\kappa_j}\frac{d}{dx}\psi_j(x; \kappa_j, \tau_j)\,. \tag{1.16}$$

La relación (1.14) puede ser representada en forma equivalente

$$\psi[n; \lambda] = \mathbb{A}_n\psi[0; \lambda]\,, \qquad \mathbb{A}_n = A_nA_{n-1}\ldots A_1\,, \tag{1.17}$$

la cual juega un rol fundamental en el siguiente analisis. Aquí los operador diferencial de primer orden A_j son definidos recursivamente a partir de las funciones (1.13) en la forma,

$$A_1 = \psi_1 \tfrac{d}{dx} \tfrac{1}{\psi_1} = \tfrac{d}{dx} - (\ln \psi_1)_x, \tag{1.18}$$

$$A_j = (\mathbb{A}_{j-1}\psi_j) \tfrac{d}{dx} \tfrac{1}{(\mathbb{A}_{j-1}\psi_j)} = \tfrac{d}{dx} - (\ln(\mathbb{A}_{j-1}\psi_j))_x\,, \quad j = 2, \dots\,. \tag{1.19}$$

En efecto, la equivalencia de (1.17) a (1.14) para $n = 1, 2$ es probada directamente. Asumiendo que

$$\mathbb{A}_n\psi[0;\lambda] = \frac{\mathbb{W}(\psi_1, \dots, \psi_n, \psi[0;\lambda])}{\mathbb{W}(\psi_1, \dots, \psi_n)}, \tag{1.20}$$

es valido para $n > 2$, Eqs. (1.19) y (1.20) dan

$$\mathbb{A}_{n+1}\psi[0;\lambda] = A_{n+1}\left(\mathbb{A}_n\psi[0;\lambda]\right) = (\mathbb{A}_n\psi_{n+1})\frac{d}{dx}\left(\frac{1}{(\mathbb{A}_n\psi_{n+1})}\mathbb{A}_n\psi[0;\lambda]\right)\,, \tag{1.21}$$

y

$$\begin{aligned}\mathbb{A}_{n+1}\psi[0;\lambda] &= \frac{\mathbb{W}(1,\dots,n,n+1)}{\mathbb{W}(1,\dots,n)}\left(\frac{\mathbb{W}(1,\dots,n)}{\mathbb{W}(1,\dots,n,n+1)}\frac{\mathbb{W}(1,\dots,n,0)}{\mathbb{W}(1,\dots,n)}\right)_x \\ &= \frac{\mathbb{W}(\mathbb{W}(1,\dots,n,n+1),\mathbb{W}(1,\dots,n,0))}{\mathbb{W}(1,\dots,n)\mathbb{W}(1,\dots,n,n+1)},\end{aligned} \tag{1.22}$$

acá $\mathbb{W}(1,\dots,n,n+1) = \mathbb{W}(\psi_1,\dots,\psi_{n+1})$, $\mathbb{W}(1,\dots,n,0) = \mathbb{W}(\psi_1,\dots,\psi_n,\psi[0;\lambda])$. La identidad entre Wronskianos

$$\mathbb{W}(f_1,\dots,f_n,g,h)\mathbb{W}(f_1,\dots,f_n) = \mathbb{W}(\mathbb{W}(f_1,\dots,f_n,g),\mathbb{W}(f_1,\dots,f_n,h))\,, \tag{1.23}$$

es valida para cualquier elección de funciones $f_1,\dots,f_n$, g y h [39], nos permite representar la fracción (1.22) en la forma del lado derecho de (1.20) con n cambiado por $n+1$. Esto prueba la equivalencia de (1.17) a (1.14) por inducción.

La definición (1.19) y la relación (1.20) provee tambien la siguiente posible representación del operador A_n,

$$A_n = \frac{d}{dx} - (\ln \mathbb{A}_{n-1}\psi_n)_x = \frac{d}{dx} - \left(\ln\frac{\mathbb{W}_n}{\mathbb{W}_{n-1}}\right)_x \equiv \frac{d}{dx} + \mathcal{W}_n\,, \tag{1.24}$$

acá

$$\mathcal{W}_n = \Omega_n - \Omega_{n-1}\,, \qquad \Omega_n = -(\ln \mathbb{W}_n)_x\,. \tag{1.25}$$

Entonces (1.25) en conjuto con la Eq. (1.12) nos da una más util representación del potencial n-solitónico,

$$U_n = 2\Omega_{nx}\,, \tag{1.26}$$

teniendo en mente esta relación, nosotros llamaremos a Ω_n pre-potencial del sistema n-solitónico. Coherentemente con Eqs. (1.20) y (1.18), en (1.24) y (1.25) asumimos $\mathbb{W}_0 = 1$, $\Omega_0 = 0$, $V_0 = 0$, y tenemos $\mathbb{W}_1 = \cosh\kappa_1(x+\tau_1)$,

$$\Omega_1 = -\kappa_1\tanh\kappa_1(x+\tau_1), \qquad U_1 = -\frac{2\kappa_1^2}{\cosh^2\kappa_1(x+\tau_1)}\,. \tag{1.27}$$

Como sigue de (1.19), el operador diferencial de primer orden A_j aniquila la autofunción no física sin nodos $\mathbb{A}_{j-1}\psi_j$ de H_{j-1} de autovalor $-\kappa_j^2$. Por otro lado, $A_j^\dagger$ aniquila una autofunción $1/(\mathbb{A}_{j-1}\psi_j)$,el cual es el estado ligado de más baja energía de H_j con autovalor $-\kappa_j^2$. Esto significa que U_n y U_{n-1} estan relacionados por una transformación de Darboux. Explicitamente nosotros tenemos las relaciones

$$U_n = \mathcal{W}_n^2 + \mathcal{W}_{nx} - \kappa_n^2\,, \qquad U_{n-1} = \mathcal{W}_n^2 - \mathcal{W}_{nx} - \kappa_n^2\,, \tag{1.28}$$

$$A_n A_n^\dagger = H_n + \kappa_n^2\,, \qquad A_n^\dagger A_n = H_{n-1} + \kappa_n^2\,. \tag{1.29}$$

En correspondencia con (1.29), los generadores de Darboux de primer orden A_n y $A_n^\dagger$ entrelazan los sistemas n- y $(n-1)$-solitónicos,

$$A_n H_{n-1} = H_n A_n, \qquad A_n^\dagger H_n = H_{n-1} A_n^\dagger,$$

y relacionan sus autoestados,

$$\psi[n;\lambda] = A_n \psi[n-1;\lambda], \qquad A_n^\dagger \psi[n;\lambda] = (\lambda + \kappa_n^2)\psi[n-1;\lambda],$$

compare con (1.17). Por otro lado, los operadores diferenciales de orden n $\mathbb{A}_n$ and $\mathbb{A}_n^\dagger$ entrelazan H_n y el Hamiltoniano de la partícula libre H_0,

$$\mathbb{A}_n H_0 = H_n \mathbb{A}_n, \qquad \mathbb{A}_n^\dagger H_n = H_0 \mathbb{A}_n^\dagger\,. \tag{1.30}$$

Como sigue de (1.17), los estados de onda plana de la partícula libre e^{ikx} son mapiados en autofunciones de H_n de la forma $\psi_n(x,k) = P_n(x,k)e^{ikx}$, acá P_n es un polinomio de orden n en k, $H_n\psi_n(x,k) = k^2\psi_n(x,k)$. Esto significa que $U_n(x)$ es un potencial transparente o potencial de Bargmann-Kay-Moses [34], para el cual el coeficiente de transmición es facilmente computable. Para las fuciones (1.13) nosotros tenemos que $\psi_j(x) \sim e^{\pm\kappa_j(x+\tau_j)}$ para $x \to \pm\infty$. Luego encontramos que $A_j \to \frac{d}{dx} \pm \kappa_j$ para $x \to \mp\infty$, y en este limite $P_n \to P_{n\mp} = \prod_{j=1}^n (ik \pm \kappa_j)$. Para la amplitud de transmisión $t(k) = P_{n+}/P_{n-}$ esto da

$$t(k) = \prod_{j=1}^{n} \left(\frac{k + i\kappa_j}{k - i\kappa_j}\right)\,. \tag{1.31}$$

Esta clase de potenciales transparentes que consideramos tabien puede ser relacionada naturalmente con otro sistema completamente integrable, e este caso cpn la ecuacion de Schrödiger no lineal.

Para ver esto, primero mostraremos que estos potenciales transparentes $U_n(x)$ pueden ser presentados en la forma

$$U_n(x) = -4\sum_{j=1}^{n} \kappa_j \hat{\psi}_{n,j}^2(x), \tag{1.32}$$

en terminos de los estados ligados normalizados del Hamiltoniano H_n,

$$\hat{\psi}_{n,j}(x) = \mathcal{N}_j^{-1}\psi[n,-\kappa_j^2](x), \qquad \mathcal{N}_j^2 = 2\kappa_j \prod_{\ell=1,\,\ell\neq j}^{n} |\kappa_\ell^2 - \kappa_j^2|\,, \qquad \int_{-\infty}^{+\infty} \hat{\psi}_{n,j}^2(x)dx = 1\,, \quad (1.33)$$

acá es asumido que para $n = 1$ el producto en la expresión para $\mathcal{N}_1^2$ es reducido a 1. Usando la relación $\frac{d}{dx}\mathbb{W}_n = \sum_{j=1}^{n} \mathbb{W}(\psi_1,\ldots,\frac{d\psi_j}{dx},\ldots,\psi_n)$, podemos reescribir la Eq. (1.12) en la forma $U_n(x) = -2\sum_{j=1}^{n} \mathbb{W}\left(\mathbb{W}_n, \mathbb{W}(\psi_1,\ldots,\frac{d\psi_j}{dx},\ldots,\psi_n)\right)/\mathbb{W}_n^2$. La identidad entre Wronskianos (1.23) nos permite representar el potencial en una forma equivalente

$$U_n(x) = -2\sum_{j=1}^{n} \frac{\mathbb{W}(\psi_1,\ldots,\psi_j,\frac{d\psi_j}{dx},\ldots,\psi_n)\mathbb{W}(\psi_1,\ldots,\psi_{j-1},\psi_{j+1},\ldots,\psi_n)}{\mathbb{W}_n^2}\,. \quad (1.34)$$

La relación

$$\mathbb{W}(\psi_1,\ldots,\psi_j,\frac{d\psi_j}{dx},\ldots,\psi_n) = \frac{1}{2}\kappa_j\mathcal{N}_j^2\mathbb{W}(\psi_1,\ldots,\psi_{j-1},\psi_{j+1},\ldots,\psi_n), \quad (1.35)$$

acá $\mathcal{N}_j^2$ es definido en (1.33), sigue de identidades basicas del determinante. Usando esta ultima relación en conjunto con las Eqs. (1.14), (1.15) y (1.16), podemos reescribir (1.34) en terminos de los estados ligados no normalizados de H_n,

$$U_n(x) = -4\sum_{j=1}^{n} \kappa_j\mathcal{N}_j^{-2}\psi^2[n,-\kappa_j^2](x)\,. \quad (1.36)$$

Aplicando una vez más la identidad (1.23) obtenemos

$$\frac{d}{dx}\left(\frac{\mathbb{W}(\psi_1,\ldots,\frac{d\psi_j}{dx},\ldots,\psi_n)}{\mathbb{W}_n}\right) = \frac{\mathbb{W}(\psi_1,\ldots,\psi_{j-1},\psi_{j+1},\ldots,\psi_n,\psi_j,\frac{d\psi_j}{dx})\mathbb{W}(\psi_1,\ldots,\psi_{j-1},\psi_{j+1},\ldots,\psi_n)}{\mathbb{W}_n^2}\,. \quad (1.37)$$

Eq. (1.35) nos da luego $\frac{d}{dx}(\mathbb{W}(\psi_1,\ldots,\frac{d\psi_j}{dx},\ldots,\psi_n)/\mathbb{W}_n) = 2\kappa_j\mathcal{N}_j^{-2}\psi^2[n,-\kappa_j^2](x)$. Integrando esta igualdad desde $-\infty$ a $+\infty$, y usando la relación $\lim_{x\to\pm\infty}\mathbb{W}(\psi_1,\ldots,\frac{d\psi_j}{dx},\ldots,\psi_n)/\mathbb{W}_n = \pm\kappa_j$, reproducimos (1.33), y presentamos (1.36) en la forma (1.32).

Debido a la relación (1.32), la ecuación $H_n\hat{\psi}_{n,j} = -\kappa_j^2\hat{\psi}_{n,j}$ para n estados ligados normalizados puede ser presentado como un sistemas de n equaciones diferenciales ordinarias acopladas nolinealmente

$$-\hat{\psi}_{n,jxx} - 4\sum_{i=1}^{n}\kappa_i\hat{\psi}_{n,i}^2\hat{\psi}_{n,j} + \kappa_j^2\hat{\psi}_{n,j} = 0\,. \quad (1.38)$$

Introduciendo un parametro de evolución t, y deniniendo $q_j(x,t) = \exp(i\kappa_j^2 t)\hat{\psi}_{n,j}(x)$. Luego encontramos que estas funciones satisfacen el sistema de n ecuaciones de Schrödinger no lineal acopladas,

$$iq_{jt} = -q_{jxx} - 4\sum_{i=1}^{n}\kappa_i|q_i|^2q_j\,. \quad (1.39)$$

En el caso mas simple $n = 1$, esta se reduce a la ecuación de Schrödinger no lineal enfocada,

$$iq_t + q_{xx} + 4\kappa|q|^2 q = 0\,. \tag{1.40}$$

Por lo tanto, las n soluciones ligadas de la Schrödinger lineal dependiente del tiempo con potenciales transparentes n-solitónicos independientes del tiempo, proveen souciones para el sistema de n ecuaciones de Schrödinger no lineal acopladas.

1.1.2. Supersimetría exótica de pares de Hamiltonianos multi-solitónicos transparentes

En esta subsección describo cortamente una estructura supersimétrica exótica $N = 4$ que aparece entre pares de sistemas tipo Schrödinger transparentes n-solitónicos de la forma más general [5] y obserbaremos como esta supersimetría extendida esta relacionada con la jerarquía de ecuaciones integrales de Korteweg-de Vries.

Vamos a considerar dos sistemas completamente transparentes H_n y $\tilde{H}_m$ con n y m estados ligados respectivamente $n > m$ construidos usando dos conjuntos de parametros espectrales, $(\kappa_1, \ldots, \kappa_n,\ \tau_1, \ldots, \tau_n)$ y $(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_m,\ \tilde{\tau}_1, \ldots, \tilde{\tau}_m)$. Cada uno de estos Hamiltonianos puede ser entrelazados con el Hamiltoniano de la partícula libre H_0 por medio de sus correpondientes operadores de entrelazamiento de ordenes n y m, $\mathbb{A}_n$ and $\tilde{\mathbb{A}}_m$, y por sus operadores conjugados $\mathbb{A}_n^\dagger$ y $\tilde{\mathbb{A}}_n^\dagger$, respectivamente. La relación (1.30) y relaciones similares para $\tilde{H}_m$ en conjunto con la observación que $\frac{d}{dx}$ es una integral de movimiento para la partícula libre nos permiten construil el conjunto de operadores que entrelazan a ambos sistemas transparentes H_n y $\tilde{H}_n$,

$$\mathbb{Y} = \mathbb{A}_n \tilde{\mathbb{A}}_m^\dagger\,, \qquad \mathbb{X} = \mathbb{A}_n \frac{d}{dx} \tilde{\mathbb{A}}_m^\dagger\,, \tag{1.41}$$

$$\mathbb{J}\tilde{H}_m = H_n \mathbb{J}\,, \qquad \mathbb{J}^\dagger H_n = \tilde{H}_m \mathbb{J}^\dagger, \quad \text{where} \quad \mathbb{J} = \mathbb{Y}\,,\, \mathbb{X}\,. \tag{1.42}$$

El operador $\mathbb{Y}$ es un operador diferencial de orden $n+m$, mientras que $\mathbb{X}$ es un operador diferencial de orden $n+m+1$, si uno es par el opro necesariamente es impar. Por otro lado podemos contruir integrales de movimiento analogas al momento lineal para cada uno de los hamiltonianos H_n y $\tilde{H}_n$,

$$\mathbb{Z}_n = \mathbb{A}_n \frac{d}{dx} \mathbb{A}_n^\dagger\,, \quad |\mathbb{Z}_n| = 2n+1, \qquad \tilde{\mathbb{Z}}_m = \tilde{\mathbb{A}}_m \frac{d}{dx} \tilde{\mathbb{A}}_m^\dagger\,, \quad |\tilde{\mathbb{Z}}_m| = 2m+1, \tag{1.43}$$

son los respectivos vestimientos de Darboux de la integral de la partícula libre $\frac{d}{dx}$, y so integrales para H_n y $\tilde{H}_m$ dado que es posible demostrar que

$$[\mathbb{Z}_n, H_n] = 0, \qquad [\tilde{\mathbb{Z}}_m, \tilde{H}_m] = 0. \tag{1.44}$$

El operador $\mathbb{Z}_n$ puede ser presentado en la forma $\mathbb{Z}_n = (-1)^n \frac{d^{2n+1}}{dx^{2n+1}} + \sum_{j=1}^{2n} a_{2n-j}(x) \frac{d^{2n-j}}{dx^{2n-j}}$, acá los coeficientes $a_{2n-j}(x)$ son algunanas funciones del potencial U_n y sus derivadas $U_{nx}, \ldots,$

$\frac{d^{2n-1}}{dx^{2n-1}}U_n$. La relación de conmutatividad de $\mathbb{Z}_n$ y H_n, $[\mathbb{Z}_n, H_n] = 0$, es la ecuación de Novikov, o, equivalentemente, una ecuación de alto orden dentro de la jerarquía estacionaria Korteweg-de Vries [18, 52]. En correspondencia con el teorema de Burchnall-Chaundy [37], operadores diferenciales conmutantes $\mathbb{Z}_n$ y H_n, en este caso, de ordenes mutuamente primos, $2n+1$ y 2, satisfacen identicamente una relación $\mathbb{Z}_n^2 = P_{2n+1}(H_n)$,acá $P_{2n+1}(H_n) = H_n \prod_{j=1}^{n}(H_n + \kappa_j^2)^2$ es un polinomio espectral degenerado para el sistema n-solitónico [5]. En correspondencia con esta relación $\mathbb{Z}_n$ aniquila todos los estados physicos singletes, los cuales son todos los estados ligados de energía $E_j = -\kappa_j^2$, $j = 1, \ldots, n$, y el estado $\psi[n;0] = \mathbb{A}_n 1$ de energía cero siendo el estado de energía más baja del espectro continuo de estados físicos en la zon a de dispersión, cf. Eq. (1.17). Los otros n estados aniquilados por $\mathbb{Z}_n$ son los estados no físicos de H_n de energías $E_j = -\kappa_j^2$.

En el caso más simple $n = 1$, el pre-prepotencial y el potencial estan dados por la Eq. (1.27), y tenemos que $\mathbb{Z}_1 = \frac{1}{4}\mathcal{Z}_1 + \kappa_1^2 \mathcal{Z}_0$, acá $\mathcal{Z}_0 = \frac{d}{dx}$ y $\mathcal{Z}_1 = -4\frac{d^3}{dx^3} + 6U_1\frac{d}{dx} + 3U_{1x}$ son los operadores de Lax correspondientes a las primeras dos ecuaciones de la jerarquía de equaciones de Korteweg-de Vries dependientes del tiempo , $u_t - u_x = 0$ y $u_t - 6uu_x + u_{xxx} = 0$. La relación $[\mathbb{Z}_1, H_1] = 0$ se reduce aqui a la ecuación de Novikov o condicion de involución en la forma $-\frac{1}{4}(U_{1xx} - 3U_1^2 - 4\kappa_1^2 U_1)_x = 0$, la cual se satisface debido a la igualdad

$$U_{1xx} - 3U_1^2 - 4\kappa_1^2 U_1 = 0\,, \tag{1.45}$$

valida para el potencial uno-solitónico (1.27).

En virtud de las relaciones (1.42) y (1.44), el sistema compuesto, descrito por la matriz 2×2 Hamiltoniano $\mathcal{H}_n = \text{diag}\,(H_n\,, \tilde{H}_n)$, posee seis integrales autoadjuntas de movimiento, acá hemos elegido $n = m$ por simplicidad, el caso general será estudiado más adelante

$$\mathcal{S}_1 = \begin{pmatrix} 0 & \mathbb{X} \\ \mathbb{X}^\dagger & 0 \end{pmatrix}, \quad \mathcal{Q}_1 = \begin{pmatrix} 0 & \mathbb{Y} \\ \mathbb{Y}^\dagger & 0 \end{pmatrix}, \quad \mathcal{P}_1 = -i\begin{pmatrix} \mathbb{Z}_n & 0 \\ 0 & \tilde{\mathbb{Z}}_n \end{pmatrix}, \tag{1.46}$$

y $\mathcal{S}_2 = i\sigma_3\mathcal{S}_1$, $\mathcal{Q}_2 = i\sigma_3\mathcal{Q}_1$, $\mathcal{P}_2 = \sigma_3\mathcal{P}_1$. La elexion de la matriz de Pauli diagonal σ_3 como operador de graduación $\mathbb{Z}_2$ identifica las integrales $\mathcal{S}_a$ y $\mathcal{Q}_a$, $a = 1, 2$, como operadores fermiónicos, $\{\sigma_3, \mathcal{S}_a\} = \{\sigma_3, \mathcal{Q}_a\} = 0$, mientras que $\mathcal{P}_{n,a}$ son identificados como bosónicos, $[\sigma_3, \mathcal{P}_a] = 0$. En conjunto con $\mathcal{H}_n$ ellos generan un superálgebra de Lie, en la cual el Hamitoniano $\mathcal{H}_n$ juega un rol de carga central multiplicativa. La estructura superálgebraica dada por las relaciones de anti-conmutación de estas integrales, cuya forma explicita puede ser encontrada en [5], es insensitiva a los parametros de traslación τ_j y $\tilde{\tau}_j$. Acá solo escribire la forma explicita de las relaciones de conmutación de las integrales de bosonicas con las integrales fermionicas

$$[\mathcal{P}_1, \mathcal{S}_a] = i\mathcal{H}_n\mathbb{P}_n^-(\mathcal{H}_n, \kappa\,, \tilde{\kappa})\mathcal{Q}_a, \qquad [\mathcal{P}_1, \mathcal{Q}_a] = -i\mathbb{P}_n^-(\mathcal{H}_n, \kappa, \tilde{\kappa})\mathcal{S}_{n,a}\,, \tag{1.47}$$

y los conmutadores con $\mathcal{P}_2$ tienen una forma similar pero con $\mathbb{P}_n^-(\mathcal{H}_n, \kappa, \tilde{\kappa})$ cambiado por $\mathbb{P}_n^+(\mathcal{H}_n, \kappa, \tilde{\kappa})$,

acá $\mathbb{P}_n^{\pm}(\mathcal{H}_n, \kappa, \tilde{\kappa}) \equiv \mathbb{P}_n(\mathcal{H}_n, \kappa) \pm \mathbb{P}_n(\mathcal{H}_n, \tilde{\kappa})$, y

$$\mathbb{P}_n(\mathcal{H}_n, \kappa) = \prod_{j=1}^{n} (\mathcal{H}_n + \kappa_j^2 \mathbb{1}) \,, \tag{1.48}$$

con $\mathbb{1}$ la matriz identidad 2×2. De la definición de $\mathbb{P}_n^{\pm}$ sigue que mientras $\mathbb{P}_n^{+}$ es siempre un operador de orden n en la matriz Hamiltoniano $\mathcal{H}_n$, $\mathbb{P}_n^{-}$ en caso general es un polinomio de orden $(n-1)$ en $\mathcal{H}_n$. Cabe destacar que en el caso completamente isospectral $\kappa_j = \tilde{\kappa}_j$, $j = 1, \ldots, n$, $\mathbb{P}_n^{-}$ se reduce al operador cero. Esto significa que en dicho caso completamente isospectral la integral $\mathcal{P}_1$ se transforma en un carga central de la superálgebra no lineal.

En la siguente sección estudiaremos la familia completa de potenciales que permiten la existencia de una integral bosónica analoga al momento lineal para la partícula libre, para eso estudiaremos la jerarquía de ecuaciones de KdV, su formulación de Par de Lax y sus soluciones algebro-geometricas, con el fin de poder comprender este cuadro supersimétrico extendido en forma general, además de obtener el cuadro completo de soluciones de problemas asociados, como veremos en las proximas secciones.

1.2. Jerarqía de ecuaciones de Korteweg-de Vries, Integrales de Lax-Novikov y supersimetría nolineal con graduación no local

La jerarquía de ecuaciones de Korteweg-de Vries (KdV) correponde a un conjunto de sistemas nolineales completamente integrables. El método álgebro geométrico permite encontrar soluciones cuasi-periódicas en forma de segunda derivada logarítmica de la función Theta de Riemann. Estas soluciones son conocidas como en la forma de Its-Matveev [26]. Debido a la covariancia de la ecuación de KdV bajo transformaciones de Darboux, es posible, vía transformaciones de Crum-Darboux, agregar un número arbitrario de solitones de diferentes tipos en cada una de las diferentes capas cristalinas, asociadas a cada período de la función Theta de Riemann. Los solitones generan defectos que rompen la periodicidad localmente, no así asintóticamente, ya que en la lejanía de defectos solitónicos la solución recupera la forma de It-Matveev pero ganando un desfase que depende de los datos espectrales del conjunto de defectos solitónicos. En general, vía transformaciones de Crum-Darboux, es posible construir sistemas con múltiples solitones de diversos tipos viajando en diversas estructuras cristalinas dentro de las soluciones en la forma de Its-Matveev. Por medio de la representación de par de Lax [47], estas soluciones de KdV corresponden a los potenciales de los sistemas mecánico cuánticos con Hamiltonianos de tipo Schrödinger estacionario en 1+1D completamente transparentes o con número finito de bandas y estados ligados. El género g de la función theta de Riemann corresponde con el número de bandas permitidas en el espectro del operador de Schrödinger, además de la banda de dispersión [18, 26]. Más explícitamente, el operador de Schrödinger está en el corazón del método de dispersión inversa, clásico método utilizado en la resolución de la ecuación de KdV. Potenciales asintóticamente libres y transparentes con n provee de soluciones tipo-partícula, conocidas como n-solitón, con aplicaciones en diversas áreas de la física [48, 36, 34]. La formulación de par de Lax para la jerarquía de ecuaciones estacionarias de KdV (s-KdV) describe una integral de movimiento de orden impar para cada Hamiltoniano de Schrödinger estacionario. El orden de esta integral, llamada integral de Lax-Novikov, depende del número de bandas y estados ligados en el espectro de dicho Hamiltoniano. Como ya hemos adelantado, el más simple ejemplo de sistema con integral de Lax-Novikov es la partícula libre $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \mathrm{cte}$, claramente es un potencial completamente transparente, cuyos autoestados tienen forma de onda plana y en este caso la integral de Lax-Novikov es simplemente el momento lineal $p = -i\hbar\frac{d}{dx}$, $[H, p] = 0$. En lo que sigue del texto usaremos la notación $\hbar = 2m = 1$, pero debemos recordar que al reconstruir la constante de Planck los defectos agregados a los potenciales, mediante transformaciones de Darboux y en general transformaciones de Crum-Darboux, son proporcionales a $\frac{\hbar^2}{m}$.

La jerarquía estacionaria inhomogénea de KdV [26] es definida recursivamente de la siguiente

forma

$$f_0 = 1, \quad f_{\ell,x} = -\frac{1}{4} f_{\ell-1,xxx} + u f_{\ell-1,x} + \frac{1}{2} u_x f_{\ell-1}, \tag{1.49}$$

siendo las ecuaciones de la jerarquía de KdV

$$2 f_{\ell,x} = \frac{du}{ds}, \tag{1.50}$$

acá s es un parámetro de evolución, es posible de interpretar como el tiempo según el sistema físico que se desee interpretar. Explícitamente uno encuentra

$$\begin{aligned} f_0 &= 1, \\ f_1 &= \frac{1}{2} u + c_1 \\ f_2 &= -\frac{1}{8} u_{xx} + \frac{3}{8} u^2 + c_1 \frac{1}{2} u + c_2 \\ f_3 &= \frac{1}{32} u_{xxxx} - \frac{5}{16} u u_x - \frac{5}{32} u_x^2 + \frac{5}{16} u^3 \\ &\quad + c_1 (-\frac{1}{8} u_{xx} + \frac{3}{8} u^2) + c_2 \frac{1}{2} u + c_3, \quad etc \end{aligned} \tag{1.51}$$

acá c_ℓ son constantes de integración reales.

Estas ecuaciones permiten una formulación de par de Lax, la cual permite una reinterpretación desde el estudio de simetrías en mecánica cuántica, dado que uno de los elementos del par de Lax corresponde a un operador de Schrödinger y como veremos a continuacion, soluciones estacionarias de la jerarquía de ecuaciones de KdV, $\frac{du}{ds} = 0$ nos entregan el conjunto completo de potenciales con integrales de movimiento de tipo Lax-Novikov para sistemas mecánico cuánticos en una línea El par de operadores en el par de Lax son

$$H = -\frac{d^2}{dx^2} + u, \tag{1.52}$$

y

$$P_{2g+1} = -i \sum_{\ell=1}^{g} \left(f_{g-\ell} \frac{d}{dx} - \frac{1}{2} f_{g-\ell,x} \right) H^\ell, \tag{1.53}$$

los cuales definen una relación de Lax

$$[P_{2g+1}, H] = -2i f_{g+1,x}, \tag{1.54}$$

la cual como condición de involución, reconstruye la jerarquía de ecuaciones estacionarias de KdV $2i f_{g+1,x} = \mathrm{sKdV}_g(u) = 0$, pero a la vez aseguran que P es una integral de movimiento para H. Las constantes c_ℓ que aparecen en (1.51) estan fijas en terminos de las energías E_m de los bordes del espectro de H en dependencia del potencial u,

$$c_k = - \sum_{\substack{i=j_0, j_1, \ldots, j_2 g = 0 \\ j_0 + j_1 + .. + j_2 g = k}}^{k} 2^{-2k} \prod_{i=0}^{2g} \frac{(2j_i)!}{(j_i!)^2 (2j_i - 1)} E_i^{j_i} \tag{1.55}$$

acá $k = 1, ..., n$ y $c_0 = 1$.

Sí $f_{g+1,x} = 0$ y H y P_{2g+1} son operadores hermíticos o $\mathcal{PT}$ simétricos[1] entonces (1.53) toma el rol de integral de movimiento de H. H y P_{2g+1} satisfacen la relación de Burchnall-Chaundy [37]

$$P_{2g+1}^2 = \prod_{i=0}^{2g}(H - E_i)\,, \tag{1.56}$$

la cual relaciona los autovalores y de P_{2g+1} los autovalores z de H por medio de una curva hiperelíptica

$$y^2 = \prod_{i=0}^{2g}(z - E_i)\,. \tag{1.57}$$

H y P_{2g+1} permiten un operador de graduación no local Z_2 Γ, $\Gamma A = (-1)^{|A|}A\Gamma$, $\Gamma^{\#} = \Gamma$ y $\Gamma^2 = 1$, el cual define $\mathbb{P}_1 = P_{2g+1}$ y $\mathbb{P}_2 = i\Gamma P_{2g+1}$ como un operador de graduación impar el cual describe una supersimetría nolineal dada por la relación de Burchnal-Chaundy

$$\{\mathbb{P}_a, \mathbb{P}_b\} = 2\delta_{ab}\prod_{i=0}^{2g}(H - E_i). \tag{1.58}$$

El operador Γ para potenciales simétricos puede ser remplazado por el operador de reflección, y en general también admite una representación de reflección en la coordenada y en las fases.

La fórmula de Its-Matveev [11] para los potenciales con g-bandas, está dada por

$$u(x) = -2\frac{d^2}{dx^2}\ln(\theta(x\mathbf{v} + \boldsymbol{\phi}, \tau)) + \Lambda_0, \quad \mathbf{v}, \boldsymbol{\phi} \in \mathbb{C}^g, \tag{1.59}$$

los autoestados del Hamiltoniano correspondiente a los potenciales finite-gap (1.59) están dados en la forma

$$\psi(r, x) = \frac{\theta(x\mathbf{v} + \boldsymbol{\phi} + \boldsymbol{\alpha}(r), \tau)}{\theta(x\mathbf{v} + \boldsymbol{\phi}, \tau)}\exp\left(-ix\xi(r)\right), \tag{1.60}$$

donde θ es la función Theta de Riemann de género g:

$$\theta(\mathbf{z}, \tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g}\exp\left(2\pi i < \mathbf{n}, \mathbf{z} > +\pi i < \mathbf{n}, \mathbf{n}\tau >\right), \quad \mathbf{z} \in \mathbb{C}^g, \tag{1.61}$$

$$\theta(\mathbf{z} + \mathbf{a}, \tau) = \theta(\mathbf{z}, \tau), \quad \mathbf{a} \in \mathbb{Z}^g, \tag{1.62}$$

τ es una matriz $g \times g$ simétrica con parte imaginaria definida positiva, cuyos elementos, al igual que las componentes de $\mathbf{v}$ y la constante Λ_0 son determinados en forma única por las energías de los bordes del espectro de H, mientras que $\boldsymbol{\alpha}(r)$ y $\xi(r)$ también dependen de un punto $r = (z, y)$ en la curva hiper elíptica (1.57) tal que $H\psi(r, x) = z\psi(r, x)$ y $P_{2g+1}\psi(r, x) = y\psi(r, x)$. $\boldsymbol{\phi}$ depende de todos los datos espectrales de H.

El límite en el cual las bandas se convierten en estados ligados, debe ser estudiado cuidadosamente, teniendo que elegir con precaución el período fundamental que será preservado, debido

[1] $\mathcal{P}f(x) = f(-x)\mathcal{P}$, $\mathcal{T}f(x) = f^*(x)\mathcal{T}$, bajo la $\mathcal{PT}$ simetría el Hamiltoniano puede presentar autovalores reales, al igual que en el caso de operadores Hamiltonianos hermíticos lo cual se apega a la observación empírica

a que si los máximos de dicha estructura periódica están en los bordes del período fundamental que se quiere preservar al tomar el límite señalado anteriormente, el efecto de esta banda desaparecerá, pero si se centra el máximo en un punto prefijado (módulo desfases nolineales) antes de tomar el limite, la estructura cristalina se estirará de tal forma, que el efecto resultante será el de un solitón centrado en este punto prefijado (módulo desfases nolineales). Otra interpretación de estos dos límites posibles, es desde el punto de vista del parámetro modular. En el caso más simple: el del potencial de Lamé, en el cual solo existe una banda finita permitida, una banda finita prohibida, una zona semifinita prohibidas y una zona semifinita de dispersión. La forma de este potencial y el ancho de la banda finita dependen de un parámetro llamado parámetro modular k, $0 < k < 1$. El límite del parámetro modular tiende a cero elimina la banda convirtiendo el potencial de Lamé en una partícula libre, mientras el límite uno transforma el potencial de Lamé en el potencial de Pöschl-Teller.

En general la ecuación estacionaria de KdV $2f_{g+l+1,x}(u_{g,l}(x)) = 0$ con parámetros c_ℓ, $\ell = 0, \ldots, 2g+2l$, definidos por las energías $\{E_0, ..., E_{2g}\} \cup (\cup_{i=1,\ldots,l}\{z(r_{i,1}), z(r_{i,1})\})$ tiene soluciones con $P_{2g+2l+1}$ irreducible cuando $u_{g,l}$ toma la forma

$$u_{g,l}(x) = u(x) - 2\frac{d^2}{dx^2}\ln(\mathbb{W}(\psi_{a_{1,1},a_{1,2}}(r_{1,1}, r_{1,2}, x), \ldots, \psi_{a_{l,1},a_{l,2}}(r_{l,1}, r_{l,2}, x))), \tag{1.63}$$

acá $a_i, a_2 \neq 0$, $r_{i,a} = z_i, y_{i,b}$ con $z_i \neq z_j$ y $z_i \neq z_j$, $i, j = 1, \ldots, l$ y $b = 1, 2$ son puntos en cartas opuestas relacionadas a la curva hiper-elíptica singular

$$y^2 = \prod_{i=0}^{2g}(z - E_i)\prod_{i=1}^{l}(z - z_i)^2, \tag{1.64}$$

en el caso más general $u_{g,l}(x)$ puede ser singular y de valor complejo. $u_{g,l}(x)$ en este caso define un par de Lax en la forma

$$H_{g,l} = H(u_{g,l}(x)), \quad P_{2g+2l+1}(u_{g,l}(x), \partial\sigma(H_{g,l})), \tag{1.65}$$

donde el vestimiento de Darboux de la integral de Lax-Novikov $P_{2g+2k+1}(u_{g,k}(x), \partial\sigma(H_{g,k}))$, $k < l$, $k = 1, 2, \ldots, l$ nos da la identidad

$$P_{2g+2l+1}(u_{g,l}(x), \partial\sigma(H_{g+l})) = \mathbb{A}_{l,k+1}P_{2g+2k+1}(u_{g,k}(x), \partial\sigma(H_{g,0}))\mathbb{A}^{\#}_{l,k+1}, \tag{1.66}$$

acá $u_{g,0}(x) = u(x)$, $H_{g,0} = H$, $\partial\sigma(H_{g,k}) = \{E_0, ..., E_{2g}\} \cup (\cup_{i=1,\ldots,k}\{z(r_{i,1}), z(r_{i,1})\})$ y $\mathbb{A}_{l,k} = A_l A_{l-1} \cdots A_k$ y A_l son definidos como en 1.5 pero cambiando $\psi_i \to \psi_{a_{i,1},a_{i,2}}(r_{i,1}, r_{i,2}, x)$,

Para realizar una interpretación desde el punto de vista mecánico cuántico el operador H debe cumplir el rol de Hamiltoniano y $P_{2g+2l+1}$ el rol de integral de movimiento. Para esto, debemos requerir que H y P sean operadores hermíticos sin singularidades en el eje real $x \in \mathbb{R}$. No es necesario exigir este intervalo como condición de hermiticidad (un ejemplo de esto es el pozo infinito) pero dada la naturaleza transparente de los potenciales solitónicos y la no periodicidad

de los potenciales ésta es la elección más natural para este tipo de sistemas, para esto debemos demandar que el espectro de los potenciales sea real $E_i, z_j, \Lambda_0 \in \mathbb{R}$, $i = 0, \ldots, 2g$ y $j = 1, \ldots, l$, y $u_{g+l}(x)$ no tenga sigularidades, en este caso $\#$ corresponde a la operación $\dagger$ de conjugación hermítica. Es posible que $u(x)$ y $u_{g,l}(x) - u(x)$ sean ambos no singulares o ambos singulares pero en este último caso las singularidades del primer término deben borrar las singularidades del segundo. En general esto es posible cuando $z_i \in \sigma(H)^c$ acá c corresponde al complemento, o lo que es lo mismo $\psi_{a_i,a_2}(r_{i,1}, r_{i,2}, x)$ deben ser estados no físicos H. Es necesario usar el teorema de ceros para la correcta elección de los coeficientes $a_{i,1}/a_{i,2}$ y las energías $z(r_i)$, es necesario notar que hay un número infinito de soluciones.

En este caso el superálgebra nolineal asociada al operador de graduación Z_2 nolocal Γ, $\Gamma O = (-1)^{|O|} O\Gamma$, $\Gamma^{\#} = \Gamma$, $\Gamma^2 = 1$, el cual define $\mathbb{P}_1 = P_{2g+2l+1}$ y $\mathbb{P}_2 = \Gamma P_{2g+2l+1}$como operadores fermiónicos ($\{\Gamma, \mathbb{P}_b\} = 0$) que describen la siguiente supersimetría nolineal

$$[H_{g+l}, \mathbb{P}_a] = 0, \quad \{\mathbb{P}_a, \mathbb{P}_b\} = 2\delta_{ab} \prod_{j=0}^{2g} (H_{g+l} - E_j) \prod_{j=1}^{l} (H_{g+l} - z_j)^2, \tag{1.67}$$

acá la nolinealidad coresponde a la relacion de Burchnal-Chaundy entre los operadores H_{g+l} y $\mathbb{P}_a$.

En los próximos capítulos distinguiremos entre dos distintas transformaciones de Darboux a soluciones en la forma de Its-Matveev, como interpretar la integral de Lax-Novikov como cadenas de transformaciones de Darboux y como agregar defectos solitónicos a potenciales transparentes del operador de Schrödinger y del Bogoliubov-de Gennes. Veremos como límites en los datos espectrales deforman un tipo de transformación de darboux en otra, generando reducciones de orden espontáneas en las integrales de Lax-Novikov. Esta reducción de orden produce que la supersimetrías $N = 4$ producidas por la transformación de Crum-Darboux observen transmutaciones debido a límites en los datos espectrales, produciendo rompimientos espontáneos de simetrías.

Capítulo 2

Integral de Lax-Novikov interpretada como cadenas de Darboux y supersimetría extendida $N = 4$

Para entender las cadenas de Darboux debemos definir un tipo especial de transformaciones de Darboux que preservan la forma de los potenciales en la forma de Its-Matveev. Las transformaciones de Darboux que definiremos a continuación son transformaciones iso-espectrales, por lo cual tanto el potencial inicial como el final son soluciones de la misma ecuación de la jerarquia de ecuaciones inhomogéneas y estacionarias de KdV.

Auto transformaciones de Darboux: son construidas a partir de estados de la forma $\psi(r, x)$, las cuales generan el cambio

$$u(x) = -2\frac{d^2}{dx^2}\ln(\theta(x\mathbf{v} + \boldsymbol{\phi}, \tau)) + \Lambda_0 \to \tilde{u}^r(x) = -2\frac{d^2}{dx^2}\ln(\theta(x\mathbf{v} + \tilde{\boldsymbol{\phi}}^r, \tau)) + \Lambda_0, \tag{2.1}$$

la única diferencia entre ambas soluciones de la jerarquía es un desfase no lineal en las estructuras cristalinas que generan cada banda en el espectro de H. Este desfase para cada estructura cristalina está dada por el parámetro espectral $\boldsymbol{\alpha}(r)$,

$$\delta\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}}^r - \boldsymbol{\phi} = \boldsymbol{\alpha}(r). \tag{2.2}$$

Los operadores

$$X(r) = \psi(r, x)\frac{d}{dx}\frac{1}{\psi(r, x)}, \quad X(r)^{\#} \equiv -\frac{1}{\psi(r, x)}\frac{d}{dx}\psi(r, x), \tag{2.3}$$

entrelazan

$$H = X(r)^{\#}X(r) + z(r), \quad \text{y} \quad \tilde{H}^r = X(r)X(r)^{\#} + z(r) = -\frac{d^2}{dx^2} + \tilde{u}^r(x), \tag{2.4}$$

en la forma $X(r)H = \tilde{H}^r X(r)$ y $X(r)^{\#}\tilde{H}^r = HX(r)^{\#}$, respectivamente. El potencial inicial y el potencial transformado, al ser isospectrales por construcción, deben ser soluciones de la misma ecuación de la jerarqía de KdV con exactamente los mismos coeficientes c_ℓ. De esto se deduce que $\tilde{H}^r$ debe tener una integral de Lax-Novikov $\tilde{P}^r_{2g+1} = P_{2g+1}(u(x) \to \tilde{u}(x)^r)$, tal que

$$[\tilde{P}^r_{2g+1}, \tilde{H}^r] = -2i\frac{d}{dx}f_{g+1}(\tilde{u}^r(x), \partial\sigma(H)), \tag{2.5}$$

$$X(r)P_{2g+1} = \tilde{P}^r_{2g+1}X(r), \quad \text{y} \quad {\tilde{P}^r_{2g+1}}^2 = \prod_{i=0}^{2g}(\tilde{H}^r - E_i). \tag{2.6}$$

Debemos entender como las transformaciones de Crum-Darboux en forma de cadenas de auto transformaciones de Darboux definen integrales de movimiento para $H_{g,0}$, Para simplificar la notación redefiniremos

$$X_{1,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(r)) \equiv X(r), \quad \boldsymbol{\alpha}(r) \notin \mathbb{Z}^g, \tag{2.7}$$

este operador de Darboux entrelaza $H_{g,0} = H(x\mathbf{v} + \boldsymbol{\phi})$ con $\tilde{H}^r_{g,0} = H(x\mathbf{v} + \boldsymbol{\phi} + \boldsymbol{\alpha}(r))$ en la forma

$$X_{1,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(r))H(x\mathbf{v} + \boldsymbol{\phi}) = H(x\mathbf{v} + \boldsymbol{\phi} + \boldsymbol{\alpha}(r))X_{1,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(r)), \tag{2.8}$$

y nos permite definir un n-auto Crum-Darboux operador

$$X_{n,0}(x\mathbf{v}+\boldsymbol{\phi}, \boldsymbol{\alpha}(\breve{r}_1), \ldots, \boldsymbol{\alpha}(\breve{r}_n)) = X_{1,0}(x\mathbf{v}+\boldsymbol{\phi}+\sum_{j=1}^{n-1}\boldsymbol{\alpha}(\breve{r}_j), \boldsymbol{\alpha}(\breve{r}_n)) \times \ldots \times X_{1,0}(x\mathbf{v}+\boldsymbol{\phi}, \boldsymbol{\alpha}(\breve{r}_1)), \tag{2.9}$$

el cual es un operador de orden n, que entrelaza $H_{g,0} = H(x\mathbf{v} + \boldsymbol{\phi})$ con $\breve{H}^n_{g,0} = H(x\mathbf{v} + \boldsymbol{\phi} + \sum_{j=1}^n \boldsymbol{\alpha}(\breve{r}_j))$.

Podemos definir $X_{n,0}$ como una integral de movimiento cuando $\sum_{j=1}^n \boldsymbol{\alpha}(\breve{r}_j) = \mathbf{0} + \mathbf{a}$, $\mathbf{a} \in \mathbb{Z}^g$ con $\breve{z}_j = z(\breve{r}_j) \neq \breve{z}_{j'} = z(\breve{r}_{j'})$ lo cual en general es posible solo para $n = 2l$ o $n = 2g + 2l + 1$, $l = 0, 1, 2, \ldots$. En el primer caso la integral $X_{2l,0}$ corresponde a un polinomio de orden l en el Hamiltoniano lo cual siempre es una integral de movimiento reducible a la identidad. El segundo caso la integral $X_{2g+2l+1,0}$ correspondera a la integral de Lax-Novikov multiplicada por un polinomio de orden l en el Hamiltoniano más un polinomio de orden $g + l$ en el Hamiltoniano, reducible a una combinación entre el operador de Lax-Novikov y la identidad. Es imposible la existencia de alguna otra integral tanto de orden impar como de orden par diferente a las mencionadas. Ya que de existir alguna otra integral es posible demostrar que P_{2g+1} debería ser reducible a algún $\tilde{P}_{2g'+1}$, $g' < g$, pero esto es imposible ya que no existen transformaciones de Darboux que borren o agreguen bandas permitidas.

La integral de Lax-Novikov P_{2g+1} para $H_{g,0}$ puede ser escrita en función de la cadena de auto transformaciones de Darboux $X_{2g+1,0}$ en la forma

$$\begin{aligned} P_{2g+1} = &\frac{i(-1)^{g+1}}{2}X_{2g+1,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(\breve{r}_1), \ldots, \boldsymbol{\alpha}(\breve{r}_{2g+1})) \\ &-\frac{i(-1)^{g+1}}{2}X^{\#}_{2g+1,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(\breve{r}_1), \ldots, \boldsymbol{\alpha}(\breve{r}_{2g+1})). \end{aligned} \tag{2.10}$$

El par $H_{g,0} = H(x\mathbf{v} + \boldsymbol{\phi})$ y $\breve{H}^k_{g,0} \equiv H(x\mathbf{v} + \boldsymbol{\phi} + \sum_{j=1}^k \boldsymbol{\alpha}(\breve{r}_j))$, $k \leq g$[1] con la condición $\sum_{j=1}^{2g+1} \boldsymbol{\alpha}(\breve{r}_j) = \mathbf{0}$, $\breve{z}_j \neq \breve{z}_{j'}$. tiene la siguiente base de operadores de entrelazamiento

$$\mathcal{D}_k = X_{k,0}(x\mathbf{v} + \boldsymbol{\phi}, \boldsymbol{\alpha}(\breve{r}_1), \ldots, \boldsymbol{\alpha}(\breve{r}_k)), \tag{2.11}$$

[1] in general el caso $k = g$ corresponde a cuando cada $\boldsymbol{\alpha}(\breve{r}_j)$, pertenece a diferentes bandas prohibidas

y

$$\mathcal{B}_k = X^{\#}_{2g+1-k,0}\left(x\mathbf{v} + \boldsymbol{\phi} + \sum_{j=1}^{k}\boldsymbol{\alpha}(\breve{r}_j), \boldsymbol{\alpha}(\breve{r}_{k+1}), \ldots, \boldsymbol{\alpha}(\breve{r}_{2g+1})\right), \tag{2.12}$$

las cuales cumplen las siguentes relaciones de entrelazamiento

$$\mathcal{D}_k H_{g,0} = \breve{H}^k_{g,0}\mathcal{D}_k, \quad \mathcal{D}^{\#}_k \breve{H}^k_{g,0} = H_{g,0}\mathcal{D}^{\#}_k, \tag{2.13}$$

$$\mathcal{B}_{2g+1-k} H_{g,0} = \breve{H}^k_{g,0}\mathcal{B}_{2g+1-k}, \quad \mathcal{B}^{\#}_k \breve{H}^k_{g,0} = H_{g,0}\mathcal{B}^{\#}_k, \tag{2.14}$$

y permiten una factorización no lineal de $H_{g,0}$ y $\breve{H}^k_{g,0}$ en la forma

$$\mathcal{D}^{\#}_k\mathcal{D}_k = \prod_{j=1}^{k}(H_{g,0} - \breve{z}_j), \quad \mathcal{D}_k\mathcal{D}^{\#}_k = \prod_{j=1}^{k}(\breve{H}^k_{g,0} - \breve{z}_j), \tag{2.15}$$

$$\mathcal{B}^{\#}_k\mathcal{B}_{2g+1-k} = \prod_{j=k+1}^{2g+1}(H_{g,0} - \breve{z}_j), \quad \mathcal{B}_{2g+1-k}\mathcal{B}^{\#}_k = \prod_{j=k+1}^{2g+1}(\breve{H}^k_{g,0} - \breve{z}_j). \tag{2.16}$$

$\mathcal{D}_k = X_{k,0}$, $k < g$ es únicamente determinado pero $\mathcal{B}_{2g+1-k}$ tiene algunos grados de libertad en la elección de $\breve{r}_i$, $i = k+1, \ldots, 2g+1$ esto puede ser escrito en la forma

$$\mathcal{B}_{2g+1-k} - \mathcal{B}'_{2g+1-k} = \mathcal{D}_k F_{2(g-k)}(H_{g,l}), \tag{2.17}$$

acá $\mathcal{B}'_{2g+1-k}$ es otra posible definición de $\mathcal{B}_{2g+1-k}$, cambiando $\breve{r}_i \to \breve{r}'_i$, $i = k+1, \ldots, 2g+1$, tal que $\sum_{j=1}^{k}\boldsymbol{\alpha}(\breve{r}_j) + \sum_{j=k+1}^{2g+1}\boldsymbol{\alpha}(\breve{r}'_j) = \sum_{j=1}^{2g+1}\boldsymbol{\alpha}(\breve{r}_j) = 0$ y

$$F_{2(g-k)}(H_{g,l}) \equiv F_{2(g-k)}(\breve{r}_{k+1}, \ldots, \breve{r}_{2g+1}, \breve{r}'_{k+1}, \ldots, \breve{r}'_{2g+1}, H_{g,l}), \tag{2.18}$$

es un polinomio de orden $g-k$ en $H_{g,0}$. Estas últimas identidades nos indican que el operador de entrelazamiento de orden $2g+1-k$ no esta definido únicamente y esto es debido a que cualquier adición del operador de entrelazamiento de orden k, modulo un polinomio en uno de los Hamiltonianos, sigue cumpliedo la condición de operador de entrelazamiento. Por otro lado no cualquier combinación lineal de estos operadores puede escribirse como cadenas de Darboux, para aclarar este punto observaremos, más adelante, el ejemplo más simple: el potencial de Lamé.

Usando el mismo argumento tenemos que

$$\mathcal{D}_k\mathcal{B}^{\#}_{2g+1-k} = i(-1)^g P_{2g+1}(\breve{u}^{\mathcal{D}_k}_{g,0}, \partial\sigma(\breve{H}^{\mathcal{D}_k}_{g,0})) + G_{2g}(\breve{H}^{\mathcal{D}_k}_{g,0}), \tag{2.19}$$

$$\mathcal{B}_{2g+1-k}\mathcal{D}^{\#}_k = -i(-1)^g P_{2g+1}(\breve{u}^{\mathcal{D}_k}_{g,0}, \partial\sigma(\breve{H}^{\mathcal{D}_k}_{g,0})) + G_{2g}(\breve{H}^{\mathcal{D}_k}_{g,0}), \tag{2.20}$$

$$\mathcal{B}^{\#}_{2g+1-k}\mathcal{D}_k = i(-1)^g P_{2g+1}(u_{g,0}, \partial\sigma(H_{g,0})) + G_{2g}(H_{g,0}), \tag{2.21}$$

$$\mathcal{D}^{\#}_k\mathcal{B}_{2g+1-k} = -i(-1)^g P_{2g+1}(u_{g,0}, \partial\sigma(H_{g,0})) + G_{2g}(H_{g,0}), \tag{2.22}$$

acá $G_{2g}(H_{g,0})$ es un polinomio de orden $2g$ en $H_{g,0}$ definido por la raíz de la ecuación

$$G_{2g}(H_{g,0})^2 = \prod_{j=1}^{2g+1}(H_{g,0} - \breve{z}_j)^2 - \prod_{j=0}^{2g}(H_{g,0} - E_j)^2, \tag{2.23}$$

lo cual inmediatamente nos da la identidad

$$\sum_{i=0}^{2g} E_i = \sum_{i=1}^{2g+1} \breve{z}_i, \tag{2.24}$$

la cual es de utilidad a la hora de buscar soluciones de la ecuación

$$\sum_{j=1}^{2g+1} \boldsymbol{\alpha}(\breve{r}_j) = \mathbf{0}. \tag{2.25}$$

Podemos ver como (2.19) y (2.21) explican la factorizacón del operador de Lax-Novikov en función de las cadenas de Daroux. Podemos observar como estos caminos cerrados se convierten en integrales de movimiento para sus respectivos Hamiltonianos de partida.

El más simple ejemplo para entender la relación entre integrales de Lax-Novikov y cadenas de Darboux es el caso con un brecha (1-gap) entre las bandas permitidas, el potencial asociado a este espectro es periódico y es conocido como el potencial de Lamé.

2.1. Generadores de Desplazamiento de Darboux: el caso del potencial de Lamé 1-gap

Considere un Hamiltoniano de Schrödinger en una dimensión $H(x) = -\frac{d^2}{dx^2} + V(x)$ con un potencial periódico $V(x)$. Exijamos que este operador permita una familia de generadores de desplazamientos de Darboux de primer orden $\mathcal{D}(x;\lambda) = \frac{d}{dx} + \varphi(x;\lambda)$ tal que,

$$\mathcal{D}(x;\lambda)H(x) = H(x+\lambda)\mathcal{D}(x;\lambda), \tag{2.26}$$

los cuales dependen en un parámetro continuo λ. Entonces puede ser mostrado que $V(x)$ tiene que ser un potencial de Lamé 1-gap. [12, 16]. En dependencia de la funcion $\mathrm{sn}(x|k)$ de Jacobi el Hamiltoniano de Lamé 1-gap toma la forma [19]

$$H(x) = -\frac{d^2}{dx^2} + 2k^2\mathrm{sn}^2\, x - k^2\,, \tag{2.27}$$

acá k es el parámetro modular, $0 < k < 1$, y fija un periodo real, $2\mathbf{K}$, y uno imaginario, $2i\mathbf{K}'$, para el potencial (doblemente periódico). En lo que sigue no se indicará explicitamete la dependencia de las funciones elipticas y funciones relacionadas en el parámetro modular k. la elección de la constante aditiva en el potencial fija el estado de mínima energía de la banda de valencia y de todo el sistema en cero, y el 1-gap espectro de (2.27) es $\sigma(H) = [0, k'^2] \cup [1, \infty)$, acá $0 < k' < 1$ es el parámetro modular complementario, $k'^2 = 1 - k^2$. En el limite de periodo real infinito ($k \to 1 \Rightarrow$

$2\mathbf{K} \to \infty$, $2i\mathbf{K}' \to i\pi$, $\mathrm{sn}\, x \to \tanh x$) de (2.27) corresponde al potencial transparente de Pöschl-Teller con un estado ligado es su espectro, mientras que el otro limite $k \to 0$, reduce (2.27) a la particula libre.

Para constuir el generador de desplazamientos de Darboux uno paramétrico, distcutiremos algunas propiedades del Hamiltoniano de Lamé (2.27). Las soluciones de la ecuación estacionaria $H\Psi(x) = E\Psi(x)$ son dadas por funciones de Bloch en la forma

$$\Psi_{\pm}^{\alpha}(x) = \frac{\mathrm{H}(x \pm \alpha)}{\Theta(x)} \exp\left[\mp x \mathrm{Z}(\alpha)\right] , \qquad E = \mathrm{dn}^2\alpha, \tag{2.28}$$

acá H, Θ y Z son las funciones Eta, Theta y Zeta de Jacobi [19, 15]. Bajo translaciones en un periodo estos estados transforman en la forma, they transform as

$$\Psi_{\pm}^{\alpha}(x + 2\mathbf{K}) = \exp[\mp i 2\mathbf{K}\kappa(\alpha)]\Psi_{\pm}^{\alpha}(x) , \quad \kappa(\alpha) = \frac{\pi}{2\mathbf{K}} - i\mathrm{Z}(\alpha) , \tag{2.29}$$

acá $\kappa(\alpha)$ es el cuasi-momentum. La energía E es dada aquí como una función $E(\alpha) = \mathrm{dn}^2\alpha$ de un parámetro complejo α. Estas es una función eliptica de mismo parámetro modular k, y su paralelogramo fundamental $\alpha \in \mathbb{C}$ es un rectangulo con vertices en 0, $2\mathbf{K}$, $2\mathbf{K}+2i\mathbf{K}'$, y $2i\mathbf{K}'$.En los bordes de este paralelogramo, la función $\mathrm{dn}\,\alpha$ toma valores reales o puros imagiarios, y porlotanto, E es real. Los lados verticales $\alpha = i\beta + \mathbf{K}$, $0 \le \beta \le \mathbf{K}'$, y $\alpha = i\beta$, $0 \le \beta < \mathbf{K}'$, corresponden, a la banda de valencia, $0 \le E \le k'^2$, y de conducción, $1 \le E < \infty$, respectivamente, acá el cuasi-momentum $\kappa(\alpha)$ es real. Los lados horizontales $\alpha = i\mathbf{K}' + \beta$ y $\alpha = \beta$ con $0 < \beta < \mathbf{K}$ corresponden a las bandas prohibidas $-\infty < E < 0$ y $k'^2 < E < 1$, acá $\kappa(\alpha)$ valores complejos. DEntro de las bandas permitidas, (2.28)los modos de Bloch se propagan a la izquierda (el indice superior) y a la derechas (indice inferior). En los bordes de banda estos estados se reducen a ondas estacionarias descritas por un estado periodico, $\mathrm{dn}\, x = \mathrm{dn}\,(x + 2\mathbf{K})$ ($E = 0$),y dos estados antiperiodicos, $\mathrm{cn}\, x = -\mathrm{cn}\,(x + 2\mathbf{K})$ ($E = k'^2$) y $\mathrm{sn}\, x = -\mathrm{sn}(x + 2\mathbf{K})$ ($E = 1$).

Al igual que el estado base $\mathrm{dn}\, x$ ($\alpha = \mathbf{K} + i\mathbf{K}'$, $E = 0$), los auto estados no físicos en la banda prohibida más baja $-\infty < E < 0$ son funciones sin nodos, las cuales son usados para construir generadores de Darboux uno paramétricos. Como $\mathrm{dn}\,(-u) = \mathrm{dn}\,(u+2\mathbf{K}) = -\mathrm{dn}\,(u+2i\mathbf{K}') = \mathrm{dn}\, u$, es conveniente introducir la notación $\alpha = -2\tau + i\mathbf{K}'$, y asumir que $\tau \in \mathbb{R}$ mientras se mantiene en mente que $E \to -\infty$ para $\tau \to n\mathbf{K}$, $n \in \mathbb{Z}$. Cambiando el argumento, $x \to x + \tau$, para función de onda (2.28) con el indice de superior nosotros obtenemos $\Psi_{+}^{-2\tau+i\mathbf{K}'}(x + \tau) = c(\tau)F(x;\tau)$, acá $c(\tau)$ es una constante multiplicativa distinta de cero e independiente de x y

$$\frac{\Theta(x - \tau)}{\Theta(x + \tau)} \exp[x\mathfrak{z}(\tau)] \equiv F(x;\tau) , \tag{2.30}$$

acá

$$\mathfrak{z}(\tau) = \mathrm{Z}(2\tau + i\mathbf{K}') + i\frac{\pi}{2\mathbf{K}} = \mathrm{Z}(2\tau) + \frac{\mathrm{cn}\, 2\tau \mathrm{dn}\, 2\tau}{\mathrm{sn}\, 2\tau} \tag{2.31}$$

es, asta un factor $-i$, el cuasi-momentum de el estado de Bloch (2.30), $\mathfrak{z}(\tau) = -i\kappa(-2\tau + i\mathbf{K}')$,

que es una función impar en τ. La función $F(x;\tau)$ es cuasi-periodica en x, $F(x+2\mathbf{K};\tau) = \exp[2\mathbf{K}\mathfrak{z}(\tau)]F(x;\tau)$, periodica en τ, $F(x;\tau+2\mathbf{K}) = F(x;\tau)$.

Consideremos ahora el operador diferencial de primer orden

$$\mathcal{D}(x;\tau) = F(x;\tau)\frac{d}{dx}\frac{1}{F(x;\tau)} = \frac{d}{dx} - \Delta(x;\tau), \qquad \mathcal{D}^\dagger(x;\tau) = -\mathcal{D}(x;-\tau)\,, \tag{2.32}$$

cuyo modo cero es $F(x;\tau)$, $\mathcal{D}(x;\tau)F(x;\tau) = 0$. La función $\Delta(x;\tau) = F\,'(x;\tau)/F(x;\tau)$, $F\,'(x;\tau) = \frac{\partial}{\partial x}F(x;\tau)$, toma la forma

$$\begin{aligned}\Delta(x;\tau) &= \mathfrak{z}(\tau) + \mathrm{Z}(x-\tau) - \mathrm{Z}(x+\tau) \\ &= \frac{\mathrm{cn}\,2\tau\mathrm{dn}\,2\tau}{\mathrm{sn}\,2\tau} + k^2\mathrm{sn}\,2\tau\mathrm{sn}(x-\tau)\mathrm{sn}(x+\tau)\,.\end{aligned} \tag{2.33}$$

Esta obedece la ecuación de Riccati

$$\Delta^2(x;\tau) \pm \Delta'(x;\tau) = 2k^2\mathrm{sn}^2(x\pm\tau) - k^2 + \varepsilon(\tau)\,, \tag{2.34}$$

acá

$$\varepsilon(\tau) = -E(-2\tau + i\mathbf{K}') = \mathrm{cn}^2 2\tau/\mathrm{sn}^2 2\tau\,. \tag{2.35}$$

Otra relación importante es la combinación lineal de tres terminos en la siguiente forma,

$$\begin{aligned}&\Delta(x;\tau) + \Delta(x+\tau+\lambda;\lambda) + \Delta(x+\lambda;-\tau-\lambda) \\ &= \mathfrak{z}(\tau) + \mathfrak{z}(\lambda) + \mathfrak{z}(-\tau-\lambda) \equiv g(\tau,\lambda)\,,\end{aligned} \tag{2.36}$$

es independiente de x. La función $g(\tau,\lambda)$ posee las propiedades de simetría $g(\tau,\lambda) = g(\lambda,\tau) = g(\tau,-\lambda-\tau) = -g(-\tau,-\lambda)$ y puede ser presentada en la forma

$$g(\tau,\lambda) = \frac{1 - \mathrm{cn}\,2\tau\,\mathrm{cn}\,2\lambda\,\mathrm{cn}\,2(\tau+\lambda)}{\mathrm{sn}\,2\tau\,\mathrm{sn}\,2\lambda\,\mathrm{sn}\,2(\tau+\lambda)}\,. \tag{2.37}$$

A partir de la ecuación de Riccati (2.34), los operadores (2.32) factorizan los Hamiltonianos Lamé (2.27),

$$\mathcal{D}^\dagger(x;\tau)\mathcal{D}(x;\tau) = H(x+\tau) + \varepsilon(\tau)\,, \qquad \mathcal{D}(x;\tau)\mathcal{D}^\dagger(x;\tau) = H(x-\tau) + \varepsilon(\tau)\,, \tag{2.38}$$

con (2.35) jugando el rol de constante de factorización.

El cambio $\tau \to -\tau$ en la primera relación de (2.38) y el subsecuente cambio de $x \to x + 2\tau$ transforma la primera factrización en una forma equivalente $\mathcal{D}(x+2\tau;\tau)\mathcal{D}^\dagger(x+2\tau;\tau) = H(x+\tau) + \varepsilon(\tau)$, que essolo la segunda relación en (2.38) con el argumento x desplazado en 2τ.

A partir de (2.38) sigue que (2.32) son los generadores de desplazamiento de Darboux buscados

$$\mathcal{D}(x;\tau)H(x+\tau) = H(x-\tau)\mathcal{D}(x;\tau)\,, \qquad \mathcal{D}^\dagger(x;\tau)H(x-\tau) = H(x+\tau)\mathcal{D}^\dagger(x;\tau). \tag{2.39}$$

como $\mathcal{D}^\dagger(x;\tau) = -\mathcal{D}(x;-\tau)$, es suficiente considerar solo la primera relación de entrelazamiento en (2.39) mientras la segunda sigue de ésta mediante el simple cambio $\tau \to -\tau$.

2.2. Cadenas de Darboux entre sistemas de Lamé 1-gap

En esta sección construiremos generalizaciones de ordenes altos de generadores de desplazamientos de Darboux, a lo cual nos referiremos como cadenas de Darboux entre sistemas de Lamé 1-gap.

El desplazamiento mutuo entre los dos sistemas en (2.39) es 2τ mientras su cordenada promedio es x. Para una generalización es conveniente dar a cada sistema su propio parámetro de desplazamiento, para esto introduciremos la notación

$$\tau_{ab} = \frac{1}{2}(\tau_b - \tau_a) = -\tau_{ba}\,, \qquad x_{ab} = x + \frac{1}{2}(\tau_a + \tau_b) = x_{ba}. \tag{2.40}$$

Entonces $x_{ab} + \tau_{ab} = x + \tau_b$, $x_{ab} - \tau_{ab} = x + \tau_a$, y las relaciones (2.38), (2.39) pueden ser presentadas en la forma

$$\mathcal{D}_{ab}\mathcal{D}_{ab}^{\dagger} = -\mathcal{D}_{ab}\mathcal{D}_{ba} = H_a + \varepsilon_{ab}, \tag{2.41}$$

$$\mathcal{D}_{ab}H_b = H_a\mathcal{D}_{ab}, \tag{2.42}$$

acá se han introducido las siguientes notaciones

$$\mathcal{D}_{ab} = \mathcal{D}(x_{ab};\tau_{ab}) = -\mathcal{D}_{ba}^{\dagger}, \qquad H_a = H(x+\tau_a), \qquad \varepsilon_{ab} = \varepsilon(\tau_{ab}) = \varepsilon_{ba}\,, \tag{2.43}$$

ver Fig. 2.1. Acá el superpotencial Δ, el generador de desplazamiento de Darboux $\mathcal{D}_{ab}$ y la constante de factorización ε_{ab} divergen para $\tau_{ab} = n\mathbf{K}$, $n \in \mathbb{Z}$, por lo que se debe suponer $\tau_{ab} \neq n\mathbf{K}$.

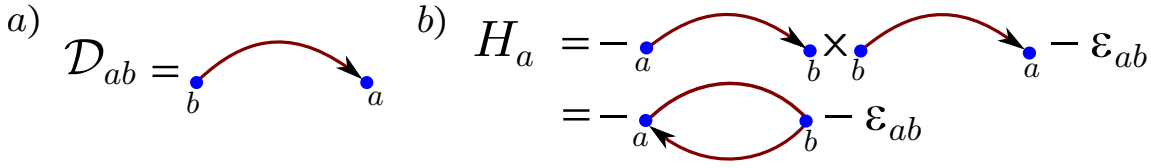

Figura 2.1: a) El generador de desplazamiento de Darboux $\mathcal{D}_{ab}$ transforma los estados de H_b en los del sistema trasladado H_a, ver (2.42). b) El Hamiltoniano H_a como una cadena cerrada de dos desplazamientos de Darboux (2.41).

Haciendo uso de la relación (2.42), uno puede encontrar el operador de segundo orden

$$\mathcal{B}_{ab/\lambda} = \mathcal{D}_{a\lambda}\mathcal{D}_{\lambda b}^{\dagger} = -\mathcal{D}_{a\lambda}\mathcal{D}_{\lambda b}\,, \qquad \mathcal{B}_{ab/\lambda}^{\dagger} = \mathcal{B}_{ba/\lambda}\,, \tag{2.44}$$

tal como para τ_{ab}, asumimos que $\tau_{a\lambda}$, $\tau_{\lambda b} \neq n\mathbf{K}$. Como el operador de primer orden $\mathcal{D}_{ab}$, este operador entrelaza los mismos Hamiltonianos H_a y H_b,

$$\mathcal{B}_{ab/\lambda}H_b = H_a\mathcal{B}_{ab/\lambda}, \tag{2.45}$$

mediante una cadena de dos desplazamientos, $\mathcal{B}_{ab/\lambda}H_b = -\mathcal{D}_{a\lambda}\mathcal{D}_{\lambda b}H_b = -\mathcal{D}_{a\lambda}H_\lambda\mathcal{D}_{\lambda b} = -H_a\mathcal{D}_{a\lambda}\mathcal{D}_{\lambda b} = H_a\mathcal{B}_{ab/\lambda}$. En esta cadena, aparece un sistema intermediario H_λ, el cual desde el punto de vista

del par de sistemas iniciales H_a y H_b es de naturaleza virtual o auxiliar. Para remarcar la naturaleza virtual de este parámetro de desplazamiento λ, se señala de una forma especial (con un slash) en la notación del operador de entrelazamiento de tipo Crum-Darboux de segundo orden $\mathcal{B}$. Para (2.44) encontramos la siguiente relación

$$\mathcal{B}_{aa/\lambda} = \mathcal{D}_{a\lambda}\mathcal{D}^{\dagger}_{a\lambda} = H_a + \varepsilon_{a\lambda}\,. \tag{2.46}$$

a diferencia de $\mathcal{D}_{ab}$, $\mathcal{B}_{ab/\lambda}$ es bien definido (para $\tau_{a\lambda}$, $\tau_{\lambda b} \neq n\mathbf{K}$) en el caso cuando $\tau_{ab} = n\mathbf{K}$. El parámetro virtual en el lado derecho en (2.46) aparece solo en el termino aditivo. We also have

$$\mathcal{B}_{ab/\lambda}\mathcal{B}^{\dagger}_{ab/\lambda} = \mathcal{B}_{ab/\lambda}\mathcal{B}_{ba/\lambda} = (H_a + \varepsilon_{a\lambda})(H_a + \varepsilon_{b\lambda})\,. \tag{2.47}$$

Haciendo uso de la constante (2.37), encontramos que una convinación lineal entre el operador de segundo orden, $\mathcal{B}$, y el de primer orden, $\mathcal{D}$, nos ofrece un operador independiente del parámetro virtual λ,

$$\mathcal{Y}_{ab} = -\mathcal{B}_{ab/\lambda} - g_{ab\lambda}\mathcal{D}_{ab}\,, \qquad \mathcal{Y}^{\dagger}_{ab} = \mathcal{Y}_{ba}\,, \tag{2.48}$$

acá se ha introducido la notación

$$g_{ab\lambda} \equiv g(\tau_{ab}, \tau_{\lambda a}) = \mathfrak{z}(\tau_{ab}) + \mathfrak{z}(\tau_{b\lambda}) + \mathfrak{z}(\tau_{\lambda a})\,. \tag{2.49}$$

LA forma Explicita del operador de entrelazamiento $\mathcal{Y}_{ab} = \mathcal{Y}(x_{ab}; \tau_{ab})$, $\mathcal{Y}_{ab}H_b = H_a\mathcal{Y}_{ab}$, es dada por

$$\mathcal{Y}(x;\tau) = \frac{d^2}{dx^2} - \Delta(x;\tau)\frac{d}{dx} - k^2\mathrm{sn}^2(x+\tau) + \mathrm{sn}^{-2}2\tau\,. \tag{2.50}$$

A partir de (2.48) obtenemos tambien que

$$\mathcal{B}_{ab/\lambda} = \mathcal{B}_{ab/\mu} + (g_{ab\mu} - g_{ab\lambda})\mathcal{D}_{ab}\,, \tag{2.51}$$

lo que corresponde a cambiar el parámetro de desplazamiento virtual, ver Fig. 2.2.

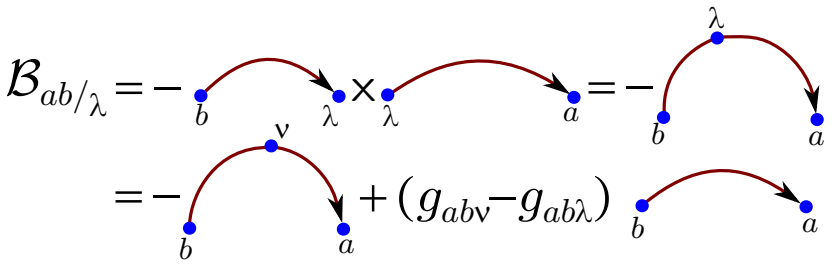

Figura 2.2: El operador de entrelazamiento de segundo orden como una cadena abierta de dos generadores de desplazamientos de primer orden; la segunda linea corresponde al cambio del parámetro virtual $\lambda \to \nu$ (2.51) más un termino proporcional al generador de desplazamiento de primer orden.

Los operadores de entrelazamiento de primer y segundo orden nos permite construir una integral no trivial para el operador Hamiltoniano H_a,

$$\mathcal{P}_a = \mathcal{D}_{ab}\mathcal{Y}_{ba} + \varepsilon_{ab}\mathcal{C}_{ab} = \mathcal{Y}_{ab}\mathcal{D}_{ba} - \varepsilon_{ab}\mathcal{C}_{ab}\,, \tag{2.52}$$

$[\mathcal{P}_a, H_a] = 0$,acá $\mathcal{P}_a = \mathcal{P}(x+\tau_a)$, $\mathcal{C}_{ab} = \mathcal{C}(\tau_{ab}) = -\mathcal{C}_{ba}$, $\mathcal{C}(\tau) \equiv g(\tau, \frac{1}{2}\mathbf{K}) = \mathrm{dn}\,2\tau/(\mathrm{sn}\,2\tau\mathrm{cn}\,2\tau)$. La integral (2.52) no es otra cosa que el operador de Lax-Novikov para el potencial de Lamé 1-gap (2.27), cuya forma explicita esta dada por

$$\mathcal{P}(x) = \frac{d^3}{dx^3} + (1 + k^2 - 3k^2\mathrm{sn}^2 x)\frac{d}{dx} - 3k^2\mathrm{sn}\,x\mathrm{cn}\,x\mathrm{dn}\,x\,, \qquad \mathcal{P}^\dagger = -\mathcal{P}\,. \tag{2.53}$$

La relación (2.52) puede ser presentada como una cadenca cerrada de tres generadores de desplazamiento de Darboux

$$\mathcal{D}_{ab}\mathcal{D}_{bc}\mathcal{D}_{ca} = \mathcal{P}_a - g_{abc}H_a + \varepsilon_{ab} - \xi_{abc}\,, \tag{2.54}$$

acá

$$\xi_{abc} \equiv \varepsilon_{ab}(g_{abc} - \mathcal{C}_{ab})\,. \tag{2.55}$$

Haciendo uso de una representación equivalente de (2.36), (2.37),

$$g(\tau;\mu) = \frac{\mathcal{C}(\tau)\varepsilon(\tau) - \mathcal{C}(\mu)\varepsilon(\mu)}{\varepsilon(\tau) - \varepsilon(\mu)}\,,$$

uno puede chequear que el objeto de tres indices (2.55) posee las misma atisimetría e los indices que g_{abc}, $\xi_{abc} = -\xi_{bac} = -\xi_{acb}$. Podemos entonces escribir

$$\mathcal{P}_a = -\mathcal{B}_{ab/c}\mathcal{D}_{ba} + g_{abc}H_a + \xi_{abc} = -\mathcal{D}_{ab}\mathcal{B}_{ba/c} - g_{abc}H_a - \xi_{abc}\,, \tag{2.56}$$

ver Fig. 2.3. Con la ayuda de (2.56) y las relaciones

$$\begin{aligned} \varepsilon_{ab}\varepsilon_{bc}\varepsilon_{ca} - \xi_{abc}^2 &= 0, \\ \varepsilon_{ab}\varepsilon_{ac} + \varepsilon_{ab}\varepsilon_{bc} + \varepsilon_{ac}\varepsilon_{bc} - 2g_{abc}\xi_{abc} &= k'^2, \\ \varepsilon_{ab} + \varepsilon_{ac} + \varepsilon_{bc} - g_{abc}^2 &= -(1 + k'^2)\,, \end{aligned} \tag{2.57}$$

encontramos que la integral de Lax-Novikov y el Hamiltoniano satisface una relacion de Burchnall-Chaundy en la forma

$$-\mathcal{P}^2 = P(H)\,, \qquad P(H) = H(H - k'^2)(H - 1)\,, \tag{2.58}$$

El operador de Lax-Novikov distingue los estados de Bloch que se mueven a la Izquierda- (Ψ_+^α) y a la derecha- (Ψ_-^α), y aniquila los estados de bordes de banda.

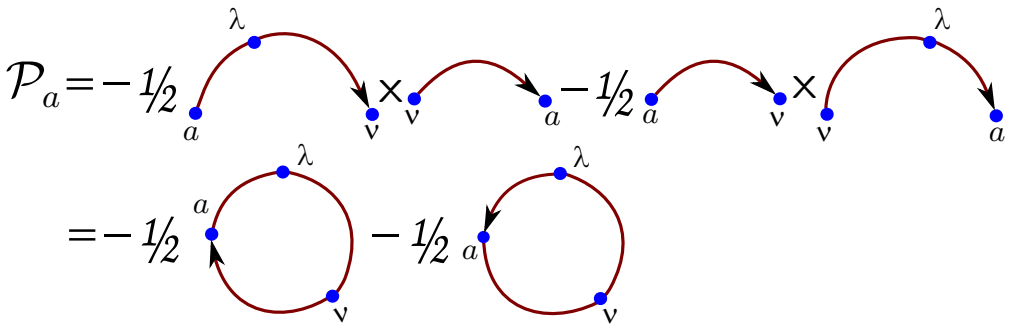

Figura 2.3: Dos representaciones para la integral de Lax-Novikov como convinacion lineal entre cadenas cerradas de Darboux: $\mathcal{P}_a = -\frac{1}{2}\mathcal{D}_{a\nu}\mathcal{B}_{\nu a/\lambda} - \frac{1}{2}\mathcal{B}_{a\nu/\lambda}\mathcal{D}_{\nu a} = -\frac{1}{2}\mathcal{D}_{a\nu}\mathcal{D}_{\nu\lambda}\mathcal{D}_{\lambda a} - \frac{1}{2}\mathcal{D}_{a\lambda}\mathcal{D}_{\lambda\nu}\mathcal{D}_{\nu a}$.

2.2.1. Imposibilidad de cadenas no triviales de orden superior

El operador de entrelazamiento de segundo orden puede tratar de ser generalizado a un operador de entrelazamiento de tercer orden. haciendo uso de la relación (2.51), uno encuentra

$$\begin{aligned}\mathcal{D}_{ab}\mathcal{D}_{bc}\mathcal{D}_{cd} &= -\mathcal{B}_{ac/b}\mathcal{D}_{cd} = -\mathcal{D}_{ab}\mathcal{B}_{bd/c} \\ &= -(H_a + \varepsilon_{dc})\mathcal{D}_{ad} + (g_{acd} - g_{acb})\mathcal{B}_{ad/c} = -(H_a + \varepsilon_{ab})\mathcal{D}_{ad} + (g_{bda} - g_{bdc})\mathcal{B}_{ad/b}\,. \end{aligned} \tag{2.59}$$

En el siguiente, el operador de entrelazamientro de cuarto orden, obtenemos de forma similar

$$\begin{aligned}\mathcal{D}_{ab}\mathcal{D}_{bc}\mathcal{D}_{cd}\mathcal{D}_{df} &= \mathcal{B}_{ac/b}\mathcal{B}_{cf/d} = (H_a + \varepsilon_{dc} + g_{adc} - g_{adf})\mathcal{B}_{af/d} \\ &+ (g_{acd} - g_{acb})(H_a + \varepsilon_{fd})\mathcal{D}_{af}. \end{aligned} \tag{2.60}$$

Como vemos en las ultimas dos relaciones es imposible crear operadores de entrelazamiento nuevos siempre estos se reducen a $\mathcal{B}$ y a $\mathcal{D}$ multiplicados por polinomios en uno de los Hamiltonianos.

Tomando en la ultima relación $f = a$, formamos una cadena cerrada de orden cuatro, la cual corresponde a una integral de movimiento para H_a, podriamos creer que esta sera una nueva integral para H_a pero un tratamiento riguroso nos da que

$$\mathcal{D}_{ab}\mathcal{D}_{bc}\mathcal{D}_{cd}\mathcal{D}_{da} = (H_a + \varepsilon_{dc})(H_a + \varepsilon_{ad}) + (g_{acb} - g_{acd})(\mathcal{P}_a + g_{acd}H_a + \xi_{acd})\,, \tag{2.61}$$

lo cual se reduce a un polinomio en H_a y a la integral de Lax-Novicov, por lo que no generamos nada nuevo.

Una cadena de Darboux de orden n cerrada de la forma $\mathcal{D}_{ab_1}\mathcal{D}_{b_1b_2}\ldots\mathcal{D}_{b_{n-1}a}$ siempre será una integral para H_a y tomará la forma

$$\mathcal{D}_{ab_1}\mathcal{D}_{b_1b_2}\ldots\mathcal{D}_{b_{n-1}a} = h_1(H_a) + h_2(H_a)\mathcal{P}_a\,, \tag{2.62}$$

acá $h_{1,2}(H_a)$ son ciertos polinomios de H_a. Mientras que cadenas abiertas de orden n de generadores de desplazamientos entre H_a y H_c, se reduciran, analogamete, a combinaciones lineales

entrelos operadores de entrelazamiento de primer y segundo orden cuyos coeficientes estaran dados por polinomios e uno de los Hamiltonianos

$$\mathcal{D}_{ab_1}\mathcal{D}_{b_1b_2}\dots\mathcal{D}_{b_{n-1}c} = f_1(H_a)\mathcal{D}_{ac} + f_2(H_a)\mathcal{B}_{ac/b_{n-1}} = \mathcal{D}_{ac}f_1(H_c) + \mathcal{B}_{ac/b_{n-1}}f_2(H_c)\,. \qquad (2.63)$$

El indice b_{n-1} en $\mathcal{B}_{ac/b_{n-1}}$ puede ser cambiado por cualquier otro indice intermediario aplicando la identidad (2.51).

se concluye entonces que cadenas abiertas de cadenas de desplazamientos de Darboux se reducen en dos ploques: el primero proporcional, al operador de entrelazamiento y generador de desplazamiento de primer orden $\mathcal{D}$, y el segundo proporcional al operador de entrelazamiento construido a partir de la composición de dos generadores de desplazamiento de Darboux, $\mathcal{B}$, cuyos coeficientes son ciertos polinomios en el operador Hamiltoniano. En el caso de cadenas cerradas, estas se reducen siempre a funciones lineales en la integral de Lax-Novikov de tercer orden cuyos coeficientes son ciertos polinomios en el operador Hamiltoniano. Ninguna estructura nueva aparece en este conjunto de operadores. Reescribir las integrales de Lax-Novikov en funcion de Cadenas de transformaciones de Darboux Nos permite entender la estructura supersimétrica no lineal asociada a los potenciales finite-gap

Para el potencial de Lamé hemos comprendido la cercana relación entre el fenómeno del desplazamiento de Darboux [16] y la existencia de integrales de tipo Lax-Novikov e integrales de entrelazamiento de órdenes superior, pudiendo comprender como cadenas de desplazamientos de Darboux generan integrales de movimiento o operadores de entrelazamiento irreducibles[41, 7]. La siguiente publicación corresponde al estudio de las simetrías de un par de Hamiltonianos de Lamé desplazados, en especial el estudio de la supersimetría extedida $N = 4$ para pares de potenciales de Lamé.

En el siguiente trabajo se estudian por completo las propiedades del potencial de Lamé 1-gap, su problema espectral, y el problema espectral asociado a sus integrales fermionicas de primer orden. Tambien es abordado el estudio del Rompimiento espotáneo de la supersimetría, sus limites no periódicos y su relación con el modelo de Gross y Neveu.

Exotic supersymmetry of the kink-antikink crystal, and the infinite period limit

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Some time ago, Thies *et al.* showed that the Gross-Neveu model with a bare mass term possesses a kink-antikink crystalline phase. Corresponding self-consistent solutions, known earlier in polymer physics, are described by a self-isospectral pair of one-gap periodic Lamé potentials with a Darboux displacement depending on the bare mass. We study an unusual supersymmetry of such a second-order Lamé system, and show that the associated first-order Bogoliubov-de Gennes Hamiltonian possesses its own nonlinear supersymmetry. The Witten index is ascertained to be zero for both of the related exotic supersymmetric structures, each of which admits several alternatives for the choice of a grading operator. A restoration of the discrete chiral symmetry at zero value of the bare mass, when the kink-antikink crystalline condensate transforms into the kink crystal, is shown to be accompanied by structural changes in both of the supersymmetries. We find that the infinite period limit may or may not change the index. We also explain the origin of the Darboux-dressing phenomenon recently observed in a nonperiodic self-isospectral one-gap Pöschl-Teller system, which describes the Dashen, Hasslacher, and Neveu kink-antikink baryons.

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I. INTRODUCTION

The Gross-Neveu (GN) model [1–3] is a remarkable $(1+1)$-dimensional theory of self-interacting fermions that has no gauge fields or gauge symmetries, but exhibits some important features of quantum chromodynamics, namely, asymptotic freedom, dynamical mass generation, and chiral symmetry breaking [4]. It has been widely studied over the years and the richness of its properties is still astonishing. Some time ago, Thies *et al.* showed that at finite density, the ground state of the model with a discrete chiral symmetry is a kink crystal [5], while the kink-antikink crystalline phase was found in the GN model with a bare mass term [6]. Then, Dunne and Basar derived a new self-consistent inhomogeneous condensate, the twisted kink crystal in the GN model with continuous chiral symmetry [7,8]. On the other hand, the relation of the GN model with the sinh-Gordon equation and classical string solutions in AdS_3 has been observed recently [9,10].

These two classes of the results seem to be different, but both are rooted in the integrability features of the GN model, and may be related to the Bogoliubov-de Gennes (BdG) equations incorporated implicitly in its structure. It is because of these properties that the model finds many applications in diverse areas of physics. Particularly, the model has provided very fruitful links between particle and condensed matter physics, see [11–13].

The origin of the model itself may also be somewhat related to the BdG equations. We briefly discuss these equations to formulate the aim of the present paper.

The BdG equations [14] in the Andreev approximation [15] is a set of two coupled linear differential equations, which can be presented in the form of a stationary Dirac-type matrix equation,

$$\hat{G}_1\psi = \omega\psi, \qquad \hat{G}_1 = a\sigma_1\frac{1}{i}\frac{d}{dx} - \sigma_2\Delta(x). \tag{1.1}$$

The scalar field $\Delta(x)$ is determined via a self-consistency condition, which is often referred to as a gap equation. Equation (1.1) arose in the theory of superconductivity by linearizing the *nonrelativistic* energy dispersion (in the absence of magnetic field), or, equivalently, by neglecting the second derivatives of the Bogoliubov amplitudes, see [16]. A constant a is proportional there to the Fermi momentum $\hbar k_F$. In what follows, we put $a = 1$ and $\hbar = 1$.

The Lagrangian of the GN model of the N species of self-interacting fermions is

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi + \tfrac{1}{2}g^2(\bar{\psi}\psi)^2, \tag{1.2}$$

where g^2 is a coupling constant, the summation in the flavor index is suppressed, and a bare mass term $\sim m_0$, which breaks explicitly the discrete chiral symmetry $\psi \to \gamma_5\psi$ of the massless model, is included.[1] It is the two-dimensional version of the Nambu-Jona-Lasinio model [17] (with continuous chiral symmetry reduced to the discrete one). The latter is based on an analogy with superconductivity, and was introduced as a model of symmetry breaking in particle physics. There are two equivalent methods to seek solutions for the

[1]The investigation of model (1.2) is motivated in [6] by a massive nature of quarks; there, the 't Hooft limit $N \to \infty$, $Ng^2 = \text{const}$, is considered.

GN model. One of them is the Hartree-Fock approach, in which self-consistent solutions to the Dirac equation $(i\gamma^\mu\partial_\mu - \mathcal{S})\psi = 0$ are looked for, with spinor and scalar fields subject to a constraint of the form $(\mathcal{S}(x) - m_0) = -Ng^2\langle\bar{\psi}\psi\rangle$, see [4,5,18]. For static solutions, under the appropriate choice of the gamma matrices, the Dirac equation takes the form of the BdG matrix Eq. (1.1), with $\hat{G}_1$ as a single particle fermionic Hamiltonian. The condensate field $\mathcal{S}(x)$ is identified with a gap function $\Delta(x)$, while the constraint corresponds to the above-mentioned gap equation. Another approach to seek solutions for the GN model, in which the BdG equations also play a key role, is via a functional gap equation [19,20]. There, the condensate field is given by stationary points of effective action, and a connection of the GN model with integrable hierarchies can be revealed, see [7,8,20,21]. In light of this, the relation of the GN model to the sinh-Gordon equation does not seem to be so surprising as the BdG equations arise (in a slightly modified form) as an important ingredient in solving the sine-Gordon equation, see [22,23].

We now return to the BdG matrix system (1.1). By squaring, the equations decouple,

$$\hat{H}\psi = E\psi, \qquad E = \omega^2, \qquad \hat{H} = -\frac{d^2}{dx^2} + \Delta^2 - \sigma_3\Delta'. \tag{1.3}$$

From the viewpoint of the second-order system $\hat{H} = \hat{G}_1^2$, the first-order matrix operator $\hat{G}_1$ is a nontrivial integral of motion, $[\hat{H}, \hat{G}_1] = 0$. Having also an integral σ_3, $[\hat{H}, \sigma_3] = 0$, which anticommutes with $\hat{G}_1$, we obtain a pattern of supersymmetric quantum mechanics with σ_3 identified as a grading operator. Though a system of the first- and second-order Eqs. (1.1) and (1.3) was exploited in investigations on superconductivity, its superalgebraic structure, which also includes the second supercharge $\hat{G}_2 = i\sigma_3\hat{G}_1$, seems to have gone unnoticed before the theoretical discovery of supersymmetry in particle physics. Supersymmetric quantum mechanics was then developed by Witten as a toy model for studying the supersymmetry breaking in quantum field theories [24]. Later, the relation of supersymmetric quantum mechanics with Darboux transformations was noticed [25], and found many applications [26].

Braden and Macfarlane [27], and, in a broader context, Dunne and Feinberg [28], observed that the Darboux transformed, supersymmetric partner of the one-gap periodic Lamé system [29] with a zero energy ground state is described by the same potential but translated for a half period. The superpartner, therefore, also has a zero ground state. Such a system is described by unbroken supersymmetry, in which, however, the Witten index takes a zero value. For a class of supersymmetric systems with superpartner potentials of the same form the term *self-isospectrality* was coined by Dunne and Feinberg [28]. The supersymmetric Lamé system considered in [27,28] corresponds to the kink crystalline phase discussed in [5], which describes a *periodic* generalization of the Callan-Coleman-Gross-Zee kink configurations of the GN model, see [2,16,18,30]. It was known earlier as a self-consistent solution to the GN model in the context of condensed matter physics [31], see also [32–34].

The Lamé system, like nonperiodic reflectionless solutions of the GN model, belongs to a special class of the *finite-gap* systems [25,35].[2] Some time ago, it was found that such systems in an unextended case (i.e., when a second-order Hamiltonian has a single component), are characterized by a hidden, peculiar nonlinear supersymmetry [37,38]. It is associated with a corresponding Lax operator (integral), and the grading is provided there by a reflection operator. As a consequence, the supersymmetric structure of an extended system [with a matrix Hamiltonian of the form (1.3)] turns out to be much richer than that associated with only the first-order supercharges $\hat{G}_a$, $a = 1, 2$, and integral σ_3, see [39]. It has also been shown recently [40] that the self-isospectral Pöschl-Teller system (PT), which describes the Dashen-Hasslacher-Neveu kink-antikink baryons [2], is characterized by a very unusual nonlinear supersymmetric structure that admits six more alternatives for the grading operator in addition to the usual choice of σ_3. All the local and nonlocal supersymmetry generators turn out to be the Darboux-dressed integrals of a free nonrelativistic particle. Moreover, it was shown there that the associated BdG system, with the matrix operator (1.1) identified as a first-order (Dirac) Hamiltonian, possesses its own, nontrivial nonlinear supersymmetry.

In the present paper we investigate the exotic supersymmetric structure of the kink-antikink crystal of [6,31], which is a self-consistent solution of the GN model (1.2) with a real gap function $\Delta(x;\tau)$. Parameter τ is related to m_0 and controls a central gap in the spectrum of the first-order BdG Hamiltonian operator (1.1). Simultaneously, it defines a mutual displacement, 2τ, of superpartner Lamé potentials in correspondence with the structure of the second-order Schrödinger operator (1.3). One more parameter, not shown explicitly here, defines a period of the crystal. A quarter-period value of τ corresponds to the kink crystal solution of [5] for the model (1.2) with $m_0 = 0$, which was considered in [27,28]. We also study different forms of the infinite period limit applied to the supersymmetric structure. *A priori* the picture of such a limit has to be rather involved: the Darboux dressing relates the nonperiodic kink-antikink system to a free particle, while the Darboux transformations in the periodic case are expected to be just self-isospectral displacements, see [31,39,41,42].

[2]There is also the relation of the one-gap Lamé equation with the sine-Gordon equation, see [36].

The outline of the paper is as follows. In the next section, we discuss the main properties of the one-gap Lamé system. In Sec. III we construct its self-isospectral extension by employing certain eigenfunctions of the Lamé Hamiltonian. We investigate the action of the first-order Darboux displacement generators, and discuss the spectral peculiarities of the obtained supersymmetric system. Section IV is devoted to the study of the properties of a superpotential (gap function) that is an elliptic function both in a variable and a shift parameter. These properties are employed in Sec. V, where we construct the second-order intertwining operators, identify further local matrix integrals of motion, and compute a corresponding nonlinear superalgebra. In Sec. VI we show that the system possesses six more, nonlocal integrals of motion, each of which may be chosen as a $\mathbb{Z}_2$ grading operator instead of the usual integral σ_3 of the supersymmetric quantum mechanics. We discuss alternative forms of the superalgebra associated with these additional integrals and their action on the physical states of the system. In Sec. VII, we investigate a peculiar nonlinear supersymmetry of the associated first-order BdG system. Section VIII is devoted to the infinite period limit of the both, second- and first-order supersymmetric systems. In Sec. IX we clarify the origin of the Darboux-dressing phenomenon that takes place in the nonperiodic self-isospectral PT system, which was revealed in [40]. In Sec. X we discuss the obtained results. To provide a self-contained presentation, the necessary properties of Jacobi elliptic functions and of some related nonelliptic functions are summarized in the two appendices.

II. ONE-GAP LAMÉ EQUATION

In this section we discuss the properties of the Lamé system, which is necessary for further constructions and analysis.

Consider the simplest (and unique) *one-gap* periodic second-order system described by the Lamé Hamiltonian

$$H = -\frac{d^2}{dx^2} + 2k^2\mathrm{sn}^2 x - k^2. \tag{2.1}$$

An additive constant term is chosen here such that a minimal energy value (the lower edge of the valence band, see below) is zero. Potential $V(x) = 2k^2\mathrm{sn}^2 x - k^2$ is a periodic function with a real period $2\mathbf{K}$ (and a pure imaginary period $2i\mathbf{K}'$).[3] The general solution of the equation

$$H\Psi(x) = E\Psi(x) \tag{2.2}$$

is given by [29]

$$\Psi^{\alpha}_{\pm}(x) = \frac{\mathrm{H}(x \pm \alpha)}{\Theta(x)} \exp[\mp x\mathrm{Z}(\alpha)]. \tag{2.3}$$

[3]See Appendices A and B for the notations and properties we use for Jacobi elliptic and related functions.

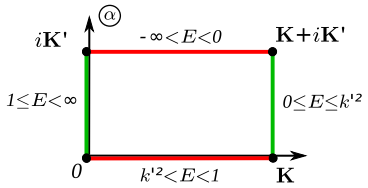

FIG. 1 (color online). The sides of the rectangle are mapped by (2.4) onto the indicated energy intervals. The vertical (horizontal) sides shown in green (red) correspond to the two allowed (forbidden) bands. Vertices $\alpha = \mathbf{K} + i\mathbf{K}'$, $\mathbf{K}$ and 0 are mapped, respectively, into the edges $E = 0$, k'^2, and 1 of the valence, $0 \le E \le k'^2$, and conduction, $1 \le E < \infty$, bands, which are described by periodic, $\mathrm{dn}x$ ($E = 0$), and antiperiodic, $\mathrm{cn}x$ ($E = k'^2$) and $\mathrm{sn}x$ ($E = 1$), functions. Vertex $i\mathbf{K}'$ as a limit point on a horizontal (vertical) side corresponds to $E = -\infty$ ($E = +\infty$).

Here H, Θ, and Z are Jacobi's Eta, Theta, and Zeta functions, and the eigenvalue $E = E(\alpha)$ is defined by the relation

$$E(\alpha) = \mathrm{dn}^2\alpha. \tag{2.4}$$

The Hamiltonian (2.1) is Hermitian, and we treat (2.2) as the stationary Schrödinger equation on a real line. We are interested in the values of the parameter α, which give real E. $\mathrm{dn}^2\alpha$ is an elliptic function with periods $2\mathbf{K}$ and $2i\mathbf{K}'$, and its period parallelogram in a complex plane is a rectangle with vertices in 0, $2\mathbf{K}$, $2\mathbf{K} + 2i\mathbf{K}'$, and $2i\mathbf{K}'$. We then look for those α in the period parallelogram for which $\mathrm{dn}\alpha$ takes real or pure imaginary values. They can be taken, for instance, on the border of the rectangle shown on Fig. 1. We have, particularly,

$$E(\mathbf{K} + i\beta) = k'^2\mathrm{cn}^2(\beta|k')\mathrm{nd}^2(\beta|k'), \quad 0 \le \beta \le \mathbf{K}', \qquad k'^2 \ge E(\mathbf{K} + i\beta) \ge 0, \tag{2.5}$$

$$E(i\beta) = \mathrm{dn}^2(\beta|k')\mathrm{nc}^2(\beta|k') = k'^2 + k^2\mathrm{nc}^2(\beta|k'), \qquad 0 \le \beta < \mathbf{K}', \qquad 1 \le E(i\beta) < \infty. \tag{2.6}$$

For (2.5) and (2.6), the eigenfunctions in (2.2) are bounded on a real line that corresponds to the two allowed (valence and conduction) bands in the spectrum. In contrast, for $\alpha = \beta$ and $\alpha = \beta + i\mathbf{K}'$, $\beta \in (0, \mathbf{K})$, a real part of $\mathrm{Z}(\alpha)$ is nonzero, and eigenfunctions (2.3) are not bounded for $|x| \to \infty$. This corresponds to the two forbidden zones, $-\infty < E < 0$ and $k'^2 < E < 1$.

Differentiation of (2.5) and (2.6) in β gives the relation

$$\frac{dE}{d\beta} = 2\eta(E)\sqrt{P(E)}, \qquad P(E) = E(E - k'^2)(E - 1). \tag{2.7}$$

The third-order polynomial $P(E)$ takes positive values inside the allowed bands, and turns into zero at their edges. $\eta(E)$ takes values -1 and $+1$ in the valence and conduction bands, respectively.

Inside the two allowed bands, (2.3) are quasiperiodic Bloch wave functions,

$$\Psi^{\alpha}_{\pm}(x+2\mathbf{K}) = e^{\mp i2\mathbf{K}\kappa(E)}\Psi^{\alpha}_{\pm}(x), \qquad \kappa(E) = \frac{\pi}{2\mathbf{K}} - i\mathrm{Z}(\alpha), \tag{2.8}$$

where the first term in quasimomentum (crystal momentum) $\kappa(E)$ originates from the imparity of the H function. In the valence, (2.5), and conduction, (2.6), bands its values are given by

$$\kappa(E(\mathbf{K}+i\beta)) = \frac{\pi}{2\mathbf{K}} - [\mathrm{Z}(\beta|k') + \frac{\pi}{2\mathbf{K}\mathbf{K}'}\beta - k'^2\mathrm{cn}(\beta|k')\mathrm{sn}(\beta|k')\mathrm{nd}(\beta|k')], \tag{2.9}$$

$$\kappa(E(i\beta)) = \frac{\pi}{2\mathbf{K}} - [\mathrm{Z}(\beta|k') + \frac{\pi}{2\mathbf{K}\mathbf{K}'}\beta - \mathrm{dn}(\beta|k')\mathrm{sn}(\beta|k')\mathrm{nc}(\beta|k')]. \tag{2.10}$$

With the help of (2.4) and (2.7), one finds a differential dispersion relation

$$\frac{d\kappa}{dE} = \eta(E)\frac{E-(\mathbf{E}/\mathbf{K})}{2\sqrt{P(E)}}, \tag{2.11}$$

where $\mathbf{E}$ is a complete elliptic integral of the second kind, see (B1). Taking into account the relation $k'^2 < \frac{\mathbf{E}}{\mathbf{K}} < 1$, see Appendix B, one finds that within both the allowed bands, quasimomentum is an increasing function of energy. It takes values 0 and $\pi/2\mathbf{K}$ at the edges $E=0$ and $E=k'^2$ of the valence band, where the Bloch-Floquet functions reduce to the periodic, dn$x$, and antiperiodic, cn$x$, functions in the real period $2\mathbf{K}$ of the system. Within the conduction band, quasimomentum increases from $\pi/2\mathbf{K}$ to $+\infty$. At the lower edge $E=1$, two functions (2.3) reduce to the antiperiodic function sn$x$. At all three edges of the allowed bands, the derivative of quasimomentum in the energy is $+\infty$. For large values of energy, $E \to +\infty$, we find that $\kappa(E) \approx \sqrt{E}$, i.e., Bloch functions (2.3) behave as the plane waves, $\Psi^{\alpha}_{\pm}(x+2\mathbf{K}) \approx e^{\mp i2\mathrm{K}\sqrt{E}}\Psi^{\alpha}_{\pm}(x)$.

Second, linear independent solutions at the edges of the allowed bands $E_i = 0, k'^2, 1$ are $\Psi_i(x) = \psi_i(x)\mathcal{I}_i$, where $\mathcal{I}_i = \int dx/\psi_i^2(x)$, and $\psi_i = \mathrm{dn}x, \mathrm{cn}x, \mathrm{sn}x$, $i=1,2,3$. The integrals are expressed in terms of a nonperiodic incomplete elliptic integral of the second kind (B2), $\mathcal{I}_1 = \frac{1}{k'^2}\mathrm{E}(x+\mathbf{K})$, $\mathcal{I}_2 = x - \frac{1}{k'^2}\mathrm{E}(x+\mathbf{K}+i\mathbf{K}')$, $\mathcal{I}_3 = x - \mathrm{E}(x+i\mathbf{K}')$. $\Psi_i(x)$ are not bounded on $\mathbb{R}$ and correspond to nonphysical states. These nonphysical solutions follow also from general solutions (2.3). For instance, $\Psi_3(x)$ may be obtained as a limit of $(\Psi^{\alpha}_{+}(x) - \Psi^{\alpha}_{-}(x))/\alpha$ as $\alpha \to 0$. Equation (2.3) provides a complete set of solutions for (2.2) as the second-order differential equation. Notice also that Bloch states (2.3) within the allowed bands are related under complex conjugation as $(\Psi^{\alpha}_{+}(x))^* = \eta\Psi^{\alpha}_{-}(x)$, where η is the same as in (2.7).

In concluding this section, we note that the function $P(E)$ in Eqs. (2.7) and (2.11) is a *spectral polynomial*. It will play a fundamental role in the nonlinear supersymmetry we discuss below.

III. SELF-ISOSPECTRAL LAMÉ SYSTEM

Consider the lower in energy E forbidden band by extending it with the edge value $E=0$ of the valence band. We introduce the notation $-2\tau + i\mathbf{K}'$ for the parameter α that corresponds to the extended interval $-\infty < E \le 0$. By taking into account relations $\mathrm{dn}(-u) = \mathrm{dn}(u+2\mathbf{K}) = -\mathrm{dn}(u+2i\mathbf{K}') = \mathrm{dn}u$, it will be convenient to not restrict the values of τ to the interval $[-\mathbf{K}/2, 0)$, but assume that $\tau \in \mathbb{R}$, while keeping in mind that $E \to -\infty$ for $\tau \to n\mathbf{K}$, $n \in \mathbb{Z}$. After a shift of the argument $x \to x+\tau$, the corresponding function Ψ^{α}_{+} from (2.3) with $\alpha = -2\tau + i\mathbf{K}'$ takes, up to an inessential multiplicative constant, the form

$$\frac{\Theta(x_-)}{\Theta(x_+)}\exp[x\mathrm{z}(\tau)] \equiv F(x;\tau), \tag{3.1}$$

where we have introduced the notations $x_+ = x+\tau$, $x_- = x-\tau$,

$$\mathrm{z}(\tau) = -i\kappa(E(-2\tau+i\mathbf{K}')) = \mathfrak{s}(\tau) + \mathrm{Z}(2\tau) = \frac{1}{2}\frac{d}{d\tau}\ln(\Theta(2\tau)\mathrm{sn}2\tau), \tag{3.2}$$

$$\mathfrak{s}(\tau) = \frac{1}{2}\frac{d}{d\tau}\ln\mathrm{sn}2\tau = \mathrm{ns}2\tau\mathrm{cn}2\tau\mathrm{dn}2\tau. \tag{3.3}$$

$F(x;\tau)$ is a quasiperiodic in x and periodic in the τ function, $F(x+2\mathbf{K};\tau) = \exp(2\mathbf{K}z(\tau))F(x;\tau)$, $F(x;\tau+2\mathbf{K}) = F(x;\tau)$. It is a regular function of τ, save for $\tau = n\mathbf{K}$, $n \in \mathbb{Z}$, [which correspond to the poles $\alpha = 2n\mathbf{K} + i\mathbf{K}'$ of dnα in (2.4)], where $F(x;\tau)$ with $x \neq 0$ undergoes infinite jumps from 0 to $+\infty$. Since $\mathrm{z}(\mathbf{K}/2) = 0$, function (3.1) reduces at $\tau = \mathbf{K}/2$ (up to an inessential multiplicative constant) to a periodic in the x function $\mathrm{dn}(x+\frac{1}{2}\mathbf{K})$, which describes a physical state with energy $E=0$ at the lower edge of the valence band of the system $H(x+\frac{1}{2}\mathbf{K})$. $F(x;\tau)$ is a nodeless function that obeys the relations $F(x;-\tau) = F(-x;\tau) = 1/F(x;\tau)$ and

$$[H(x_+) + \varepsilon(\tau)]F(x;\tau) = 0, \quad \text{where } \varepsilon(\tau) = -E(-2\tau+i\mathbf{K}') = \mathrm{cn}^2 2\tau\,\mathrm{ns}^2 2\tau. \tag{3.4}$$

A first-order differential operator is defined as

$$\mathcal{D}(x;\tau) = F(x;\tau)\frac{d}{dx}\frac{1}{F(x;\tau)} = \frac{d}{dx} - \Delta(x;\tau), \qquad \mathcal{D}^{\dagger}(x;\tau) = -\mathcal{D}(x;-\tau), \tag{3.5}$$

where

$$\Delta(x;\tau) = \frac{F'(x;\tau)}{F(x;\tau)}. \tag{3.6}$$

Operator (3.5) annihilates the function (3.1), $\mathcal{D}(x;\tau)F(x;\tau)=0$, and we find that

$$\begin{aligned}\mathcal{D}^{\dagger}(x;\tau)\mathcal{D}(x;\tau)&=H(x_+)+\varepsilon(\tau),\\ \mathcal{D}(x;\tau)\mathcal{D}^{\dagger}(x;\tau)&=H(x_-)+\varepsilon(\tau).\end{aligned}\tag{3.7}$$

By virtue of $\varepsilon(\frac{1}{2}\mathbf{K})=0$, a nonshifted Lamé Hamiltonian operator (2.1) then factorizes as $H(x)=\mathcal{D}(x+\frac{1}{2}\mathbf{K};\frac{1}{2}\mathbf{K})\mathcal{D}^{\dagger}(x+\frac{1}{2}\mathbf{K};\frac{1}{2}\mathbf{K})$. The alternative product produces a shift in the half-period $\mathbf{K}$, $H(x+\mathbf{K})=\mathcal{D}^{\dagger}(x+\frac{\mathbf{K}}{2};\frac{1}{2}\mathbf{K})\mathcal{D}(x+\frac{1}{2}\mathbf{K};\frac{1}{2}\mathbf{K})$. It is this factorization of a pair of Lamé Hamiltonians $H(x)$ and $H(x+\mathbf{K})$ that underlies the usual supersymmetric structure studied in [28] while considering the phenomenon of self-isospectrality.

Notice that while $F(x_-;\tau)$ is, up to a multiplicative constant, a nonphysical eigenfunction $\Psi_+^{-2\tau+i\mathbf{K}'}(x)$ of $H(x)$ of energy $-\varepsilon(\tau)$, function $F(x_+;-\tau)=1/F(x_+;\tau)$ coincides, up to a multiplicative constant, with another eigenfunction $\Psi_-^{-2\tau+i\mathbf{K}'}(x)$ of $H(x)$ with the same eigenvalue.

According to (3.7), the mutually shifted Hamiltonians $H(x+\tau)$ and $H(x-\tau)$ form a supersymmetric, self-isospectral periodic one-gap Lamé system

$$\mathcal{H}=\mathrm{diag}(H(x_+),H(x_-)),\tag{3.8}$$

see Fig. 2, for which $\Delta(x;\tau)$ plays the role of the superpotential, which obeys the Ricatti equations

$$\Delta^2(x;\tau)\pm\Delta'(x;\tau)=2k^2\mathrm{sn}^2(x\pm\tau)-k^2+\varepsilon(\tau).\tag{3.9}$$

Indeed, from factorizations (3.7) it follows that the $\mathcal{D}(x;\tau)$ and $\mathcal{D}^{\dagger}(x;\tau)$ intertwine the Hamiltonians $H(x_+)$ and $H(x_-)$,

$$\begin{aligned}\mathcal{D}(x;\tau)H(x_+)&=H(x_-)\mathcal{D}(x;\tau),\\ \mathcal{D}^{\dagger}(x;\tau)H(x_-)&=H(x_+)\mathcal{D}^{\dagger}(x;\tau),\end{aligned}\tag{3.10}$$

and interchange the eigenstates of the superpartner systems,

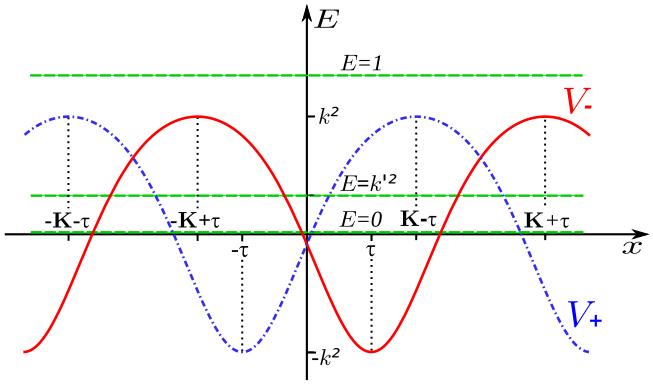

FIG. 2 (color online). The self-isospectral potentials $V_{\pm}=2k^2\mathrm{sn}(x_{\pm})-k^2$ are shown together with the edges of the valence ($0\le E\le k'^2$) and conduction ($1\le E<\infty$) bands. $V_{\pm}$ have maxima at $x=\mp\tau+(2n+1)\mathbf{K}$ and minima at $x=\mp\tau+2n\mathbf{K}$. Here $k^2=0.75$, $\mathbf{K}=2.16$, and $\tau=0.8$.

$$\begin{aligned}\mathcal{D}(x;\tau)\Psi_{\pm}^{\alpha}(x_+)&=\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,\tau)\Psi_{\pm}^{\alpha}(x_-),\\ \mathcal{D}^{\dagger}(x;\tau)\Psi_{\pm}^{\alpha}(x_-)&=-\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,-\tau)\Psi_{\pm}^{\alpha}(x_+).\end{aligned}\tag{3.11}$$

The second relation in (3.11) follows from the first one via a substitution $\tau\to-\tau$. A complex amplitude, $\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,\tau)=e^{\pm i\varphi^{\mathcal{D}}(\alpha,\tau)}\mathcal{M}^{\mathcal{D}}(\alpha,\tau)$, is given by

$$\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,\tau)=-\exp\left[\mp 2i\left(\kappa(\alpha)-\frac{\pi}{2\mathbf{K}}\right)\tau\right]\times\mathrm{ns}2\tau\frac{\Theta(2\tau\pm\alpha)\Theta(0)}{\Theta(2\tau)\Theta(\alpha)}.\tag{3.12}$$

It satisfies $(\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,\tau))^*=\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,\tau)=-\mathcal{F}_{\pm}^{\mathcal{D}}(\alpha,-\tau)$. Its modulus may be presented in the form $\mathcal{M}^{\mathcal{D}}(\alpha,\tau)=\sqrt{E(\alpha)+\varepsilon(\tau)}$, where $E(\alpha)$ for the valence and conduction bands is given by Eqs. (2.5) and (2.6). This agrees with Eq. (3.7). Notice that the modulus is even in the τ function, $\mathcal{M}^{\mathcal{D}}(\alpha,\tau)=\mathcal{M}^{\mathcal{D}}(\alpha,-\tau)$, which is nonzero except for the lower edge states of the valence band ($E=0$) in the case of $\tau=(\frac{1}{2}+n)\mathbf{K}$. A phase is well defined for $\mathcal{M}^{\mathcal{D}}\neq 0$, and satisfies the relation

$$e^{i\varphi^{\mathcal{D}}(\alpha,-\tau)}=-e^{-i\varphi^{\mathcal{D}}(\alpha,\tau)}.\tag{3.13}$$

It can be presented in the form

$$e^{i\varphi^{\mathcal{D}}(\alpha,\tau)}=-\mathrm{sign}(\mathrm{ns}2\tau)\exp\left[-2i\left(\kappa(\alpha)-\frac{\pi}{2\mathbf{K}}\right)\tau+i\varphi_{\Theta}(\alpha,\tau)\right],\tag{3.14}$$

where sign(.) is a sign function, and $\varphi_{\Theta}(\alpha,\tau)$ is a phase of $\Theta(2\tau+\alpha)$, $\varphi_{\Theta}(\alpha,\tau)=\mathrm{Im}(\int_0^{2\tau+\alpha}\mathrm{Z}(u)du)$, see Eq. (B9). Particularly, for the edge states ($i=1,2,3$), Eq. (3.12) gives $\mathcal{D}(x;\tau)\psi_i(x_+)=\mathcal{F}_i^{\mathcal{D}}(\tau)\psi_i(x_-)$, $\mathcal{D}^{\dagger}(x;\tau)\psi_i(x_-)=\mathcal{F}_i^{\mathcal{D}}(\tau)\psi_i(x_+)$, where

$$\begin{aligned}\psi_i(x)&=\mathrm{dn}x,\ \mathrm{cn}x,\ \mathrm{sn}x,\\ \mathcal{F}_i^{\mathcal{D}}(\tau)&=-\mathrm{cn}2\tau\mathrm{ns}2\tau,\ -\mathrm{dn}2\tau\mathrm{ns}2\tau,\ -\mathrm{ns}2\tau,\end{aligned}\tag{3.15}$$

and so,

$$\mathcal{M}_i^{\mathcal{D}}(\tau)=\sqrt{\varepsilon(\tau)},\quad\sqrt{k'^2+\varepsilon(\tau)},\quad\sqrt{1+\varepsilon(\tau)},\tag{3.16}$$

and $e^{i\varphi_i^{\mathcal{D}}(\tau)}=-\mathrm{sign}(\mathrm{cn}2\tau\mathrm{ns}2\tau),\ -\mathrm{sign}(\mathrm{ns}2\tau),\ -\mathrm{sign}(\mathrm{ns}2\tau)$.

As a consequence of the intertwining relations (3.10), the first-order matrix operators

$$S_1=\begin{pmatrix}0&\mathcal{D}^{\dagger}(x;\tau)\\ \mathcal{D}(x;\tau)&0\end{pmatrix},\qquad S_2=i\sigma_3S_1\tag{3.17}$$

are the integrals of motion for system (3.8). Integrals (3.17) correspond here (up to a unitary transformation of sigma matrices) to the first-order operators $\hat{G}_a$ in Sec. I. Operator $\Gamma=\sigma_3$ is a trivial integral for (3.8), $[\Gamma,\mathcal{H}]=0$, which anticommutes with S_a, $a=1,2$, $\{\Gamma,S_a\}=0$, and classifies

them as supercharges. Bosonic, $\mathcal{H}$, and fermionic, S_a, operators then satisfy the $N = 2$ supersymmetry algebra,

$$\{S_a, S_b\} = 2\delta_{ab}(\mathcal{H} + \varepsilon(\tau)), \qquad [\mathcal{H}, S_a] = 0. \quad (3.18)$$

In correspondence with (3.11) and (3.13), the eigenstates of the supercharge S_1 are

$$S_1\Psi^{\alpha}_{\pm,S_1,\epsilon} = \epsilon\mathcal{M}^{\mathcal{D}}(\alpha,\tau)\Psi^{\alpha}_{\pm,S_1,\epsilon},$$
$$\Psi^{\alpha}_{\pm,S_1,\epsilon} = \begin{pmatrix} \Psi^{\alpha}_{\pm}(x_+) \\ \epsilon e^{\pm i\varphi^{\mathcal{D}}(\alpha,\tau)}\Psi^{\alpha}_{\pm}(x_-) \end{pmatrix}, \qquad \epsilon = \pm 1. \quad (3.19)$$

Since $\varepsilon(\tau) > 0$ for $\tau \neq (\frac{1}{2} + n)\mathbf{K}$, $n \in \mathbb{Z}$, the first-order supersymmetry (3.18)[4] is dynamically broken in the general case. It is unbroken, however, for $\tau = (n + \frac{1}{2})\mathbf{K}$ by virtue of $\varepsilon((\frac{1}{2} + n)\mathbf{K}) = 0$. For these values of the shift parameter, the supercharges S_a annihilate the ground states $\mathrm{dn}(x + (n + \frac{1}{2})\mathbf{K})$ and $\mathrm{dn}(x - (n + \frac{1}{2})\mathbf{K})$ of the superpartner systems $H(x + (n + \frac{1}{2})\mathbf{K})$ and $H(x - (n + \frac{1}{2})\mathbf{K})$. Notice that with the variation of the shift parameter $\tau \neq n\mathbf{K}$, which simultaneously governs the scale of the supersymmetry breaking $\varepsilon(\tau)$, the spectrum of the second-order system (3.8) does not change. Each of its two superpartners has the same spectrum as a nonshifted Lamé system (2.1) does. Therefore, each energy level inside the valence, $0 < E < k'^2$, and conduction, $1 < E < \infty$, bands is fourth-fold degenerate in accordance with the existence of the two Bloch states, $\Psi^{\alpha}_{\pm}(x_+)$ and $\Psi^{\alpha}_{\pm}(x_-)$, of the form (2.3) for each subsystem, see Eq. (3.19). We have a two-fold degeneration at the edges $E = 0$, $E = k'^2$, and $E = 1$ of the valence and conduction bands in the spectrum of the supersymmetric system $\mathcal{H}$. Bosonic, $\Psi^{(+)}$, and fermionic, $\Psi^{(-)}$, states are defined as eigenstates of the grading operator $\Gamma = \sigma_3$, $\Gamma\Psi^{(\pm)} = \pm\Psi^{(\pm)}$, and have the general form $\Psi^{(+)} = (\Psi(x_+), 0)^T$ and $\Psi^{(-)} = (0, \Psi(x_-))^T$, where T means a transposition. In summary, we see that in both the broken and unbroken cases, the Witten index, which characterizes the difference between the number of bosonic and fermionic zero modes, is the same and equals zero.

For $\tau \neq (\frac{1}{2} + n)\mathbf{K}$ [when $\varepsilon(\tau) \neq 0$], supersymmetric relations (3.18) look different from the usual form of superalgebra in supersymmetric quantum mechanics. A simple redefinition of the matrix Hamiltonian (3.8), $\mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{H} + \varepsilon(\tau)$, will correct the form of superalgebraic relations, but will not change the conclusions on the broken (for $\tau \neq (\frac{1}{2} + n)\mathbf{K}$) form of the supersymmetric structure that we have analyzed. We shall return to this point in the discussion of the peculiar supersymmetry of the first-order Bogoliubov-de Gennes system in Sec. VII.

The described degeneracy of the energy levels in both the broken and unbroken cases is unusual for $N = 2$ supersymmetry. We will show that additional nontrivial integrals of motion may be associated with this peculiarity of the self-isospectral supersymmetric system (3.8). To identify such integrals, in the next section we investigate the function $\Delta(x;\tau)$ in greater detail.

[4]This refers to the order of the polynomial in $\mathcal{H}$ that appears in the anticommutator of the supercharges.

IV. SUPERPOTENTIAL

Being the logarithmic derivative of $F(x;\tau)$, see Eq. (3.6), the superpotential $\Delta(x;\tau)$ may be written with the help of (B11) and (B14) in terms of Jacobi's Z, or Θ and H functions,

$$\Delta(x;\tau) = \mathrm{z}(\tau) + \mathrm{Z}(x_-) - \mathrm{Z}(x_+) = \frac{1}{2}\frac{\partial}{\partial\tau}\ln\left(\frac{\mathrm{H}(2\tau)}{\Theta^2(x_-)\Theta^2(x_+)}\right). \quad (4.1)$$

The addition formula (B6) for the Z function gives another, equivalent representation,

$$\Delta(x;\tau) = \mathfrak{s}(\tau) + k^2\mathrm{sn}2\tau\mathrm{sn}(x_-)\mathrm{sn}(x_+). \quad (4.2)$$

Functions $\mathrm{z}(\tau)$ and $\mathfrak{s}(\tau)$ are defined in (3.2) and (3.3). Yet another useful representation for the superpotential may be derived from (4.2),

$$\Delta(x;\tau) = \frac{\mathrm{sn}x_-\mathrm{cn}x_-\mathrm{dn}x_- + \mathrm{sn}x_+\mathrm{cn}x_+\mathrm{dn}x_+}{\mathrm{sn}^2x_+ - \mathrm{sn}^2x_-}. \quad (4.3)$$

Having in mind relations (3.10), (3.7), and (3.9), in what follows we treat x as a variable and τ as a shift parameter. $\Delta(x;\tau)$ is an elliptic function in both its arguments with the same periods $2\mathbf{K}$ and $2i\mathbf{K}'$. It is *even* in x and *odd* in the τ function with respect to the points 0, K (modulo periods), $\Delta(-x;\tau) = \Delta(x;\tau)$, $\Delta(\mathbf{K} - x;\tau) = \Delta(\mathbf{K} + x;\tau)$, $\Delta(x;-\tau) = -\Delta(x;\tau)$, $\Delta(x;\mathbf{K} - \tau) = -\Delta(x;\mathbf{K} + \tau)$. It also obeys the relation $\Delta(x + \mathbf{K};\tau + \mathbf{K}) = \Delta(x - \mathbf{K};\tau + \mathbf{K}) = \Delta(x;\tau)$. In $\tau = 0$, $\mathbf{K}$ the function undergoes infinite jumps.

Being the elliptic function in x, $\Delta(x;\tau)$ obeys a nonlinear differential equation

$$\Delta'^2 = \Delta^4 + 2\delta_2(\tau)\Delta^2 + \delta_1(\tau)\Delta + \delta_0(\tau), \quad (4.4)$$

where $\delta_2(\tau) = 1 + k^2 - 3\mathrm{ns}^2 2\tau$, $\delta_1(\tau) = 8\mathrm{ns}^3 2\tau\mathrm{cn}2\tau\mathrm{dn}2\tau$, and $\delta_0(\tau) = -3\mathrm{ns}^4 2\tau + 2(1 + k^2)\mathrm{ns}^2 2\tau + k'^4$. As a consequence of (4.4), it also satisfies the nonlinear higher-order differential equations

$$\Delta'' = 2\Delta^3 + 2\delta_2(\tau)\Delta + \tfrac{1}{2}\delta_1(\tau),$$
$$\Delta''' = 2\Delta'(3\Delta^2 + \delta_2(\tau)). \quad (4.5)$$

Making use of (4.1), one finds the relation

$$\Delta(x + \tau + \lambda;\lambda) - \Delta(x + \lambda;\tau + \lambda) + \Delta(x;\tau) = g(\tau,\lambda). \quad (4.6)$$

The function $g(\tau,\lambda) = \mathfrak{s}(\tau) + \mathfrak{s}(\lambda) - \mathfrak{s}(\tau + \lambda) + k^2\mathrm{sn}2\tau\mathrm{sn}2\lambda\mathrm{sn}2(\tau + \lambda)$ has symmetry properties $g(\tau,\lambda) = g(\lambda,\tau) = g(\tau,-\lambda-\tau) = -g(-\tau,-\lambda)$, and may be written as

$$g(\tau,\lambda)=\mathrm{ns}2\tau\mathrm{ns}2\lambda\mathrm{ns}2(\tau+\lambda)\times[1-\mathrm{cn}2\tau\mathrm{cn}2\lambda\mathrm{cn}2(\tau+\lambda)]. \quad (4.7)$$

For a particular case $\lambda=\mathbf{K}/2$, to be important for nonperiodic limit,

$$g(\tau,\tfrac{1}{2}\mathbf{K})=\mathcal{C}(\tau),\qquad \mathcal{C}(\tau)=\mathrm{ns}2\tau\mathrm{nc}2\tau\mathrm{dn}2\tau. \quad (4.8)$$

Notice that $g(\tau,\lambda)$ takes nonzero values for all real values of its arguments.[5] Equation (4.6) is a kind of addition formula for elliptic function $\Delta(x;\tau)$. Differentiating (4.6) in x and using Ricatti Eqs. (3.9), we obtain the relation

$$\Delta'(x+\tau+\lambda;\lambda)-\Delta(x+\lambda;\tau+\lambda)\Delta(x+\tau+\lambda;\lambda)=-\tfrac{1}{2}(\Delta^2(x;\tau)+\Delta'(x;\tau)+\delta_2(\tau))-g(\tau,\lambda)\Delta(x;\tau)+G(\tau,\lambda), \quad (4.9)$$

where $G(\tau,\lambda)=\frac{1}{2}[1+k^2+g^2(\tau,\lambda)-\mathrm{ns}^2 2\tau-\mathrm{ns}^2 2\lambda-\mathrm{ns}^2 2(\tau+\lambda)]\equiv 0$.

In concluding this section we note that the functions $\delta_a(\tau)$, $a=0,\ 1,\ 2$ can be given a physical sense by expressing them in terms of the band edges energies and of $\varepsilon(\tau)$: $\delta_2(\tau)=-(\tilde{E}_1^2+\tilde{E}_2^2+\tilde{E}_3^2)$, $\delta_1(\tau)=-2\frac{d\tilde{E}_1}{d\tau}$, $\delta_0(\tau)=-\delta_2(\tau)-2(\tilde{E}_1\tilde{E}_2+\tilde{E}_1\tilde{E}_3+\tilde{E}_2\tilde{E}_3)$, where $\tilde{E}_i(\tau)=E_i+\varepsilon(\tau)$, $E_1=0$, $E_2=k'^2$, and $E_3=1$. Particularly, δ_1 measures a velocity with which a scale of supersymmetry breaking changes as a function of the shift parameter. Notice also that the first equation in (4.5) has the form of a modified Ginzburg-Landau equation, see [43], which corresponds here to a gap equation for the real condensate field in the kink-antikink crystalline phase in the Gross-Neveu model with a bare mass term, see [6,8]. At $\tau=(\frac{1}{2}+n)\mathbf{K}$, we have $\delta_1=0$, and the superpotential $\Delta(x)$ satisfies the nonlinear Schrödinger equation, the lowest nontrivial member of the modified Korteweg-de Vries hierarchy [44]. This homogenization of the second-order nonlinear differential equation can be associated with restoration of the discrete chiral symmetry in (1.2) at $m_0=0$.

V. HIGHER-ORDER INTEGRALS AND NONLINEAR SUPERALGEBRA

Now we are in a position to identify higher-order local intertwining operators and integrals of motion for the system $\mathcal{H}$. First, we find the second-order intertwining operators. Changing $\tau\to-\lambda$ and shifting the argument $x\to x+\tau+\lambda$ in the first relation from (3.10), we obtain

$$\mathcal{D}(x+\tau+\lambda;-\lambda)H(x+\tau)=H(x+\tau+2\lambda)\mathcal{D}(x+\tau+\lambda;-\lambda). \quad (5.1)$$

[5]It takes zero values at some complex values of the arguments, for instance, $\mathcal{C}(\frac{1}{2}\mathbf{K}\pm\frac{i}{2}\mathbf{K}')=0$.

Multiplying (5.1) by $\mathcal{D}(x+\lambda;\tau+\lambda)$ from the left, and using once again (3.10) on the right-hand side, we obtain the intertwining relation

$$\mathcal{B}(x;\tau,\lambda)H(x_+)=H(x_-)\mathcal{B}(x;\tau,\lambda). \quad (5.2)$$

It is generated by the second-order differential operator

$$\mathcal{B}(x;\tau,\lambda)=\mathcal{D}(x+\lambda;\tau+\lambda)\mathcal{D}^\dagger(x+\tau+\lambda;\lambda), \quad (5.3)$$

which is defined for $\lambda,\ \tau+\lambda\neq n\mathbf{K}$. For the adjoint operator we have $\mathcal{B}^\dagger(x;\tau,\lambda)H(x-\tau)=H(x+\tau)\mathcal{B}^\dagger(x;\tau,\lambda)$. In accordance with (5.1), the second-order intertwining operator (5.3) shifts the Hamiltonian's argument first for 2λ and then for $-2(\tau+\lambda)$. An equivalent representation of the operator (5.3) is

$$\mathcal{B}(x;\tau,\lambda)=-\mathcal{Y}(x;\tau)-g(\tau,\lambda)\mathcal{D}(x;\tau), \quad (5.4)$$

$$\mathcal{Y}(x;\tau)=\frac{d^2}{dx^2}-\Delta(x;\tau)\frac{d}{dx}-\frac{1}{2}(\Delta^2(x;\tau)+\Delta'(x;\tau)+\delta_2(\tau)),\qquad \mathcal{Y}^\dagger(x;\tau)=\mathcal{Y}(x;-\tau). \quad (5.5)$$

We have used here Eq. (4.6). So, the dependence of $\mathcal{B}(x;\tau,\lambda)$ on λ is localized only in the x-independent multiplier $g(\tau,\lambda)$, see Eq. (4.7).

From Eqs. (5.3) and (3.10), it follows that at $\tau=0$ the second-order intertwining operators $\mathcal{B}(x;\tau,\lambda)$ and $\mathcal{B}^\dagger(x;\tau,\lambda)$ reduce, up to an additive term $\varepsilon(\lambda)$, to the isospectral superpartner Hamiltonians, $\mathcal{B}(x;0,\lambda)=H(x)+\varepsilon(\lambda)$,[6] $\mathcal{B}^\dagger(x;0,\lambda)=H(x+2\lambda)+\varepsilon(\lambda)$.

Forgetting for the moment the $\tau=0$ case, from the viewpoint of the intertwining relation (5.2), one could conclude that the parameter λ has a "gaugelike," nonobservable nature. Such a conclusion, however, is not correct. We will return to this point later.

Since $g(\tau,\lambda)$ is nonzero for real τ and λ, operator $\mathcal{Y}(x;\tau)$, unlike $\mathcal{B}(x;\tau,\lambda)$, is not factorizable in terms of our first-order intertwining operators (with real shift parameters).[7] Nevertheless, it is the second-order intertwining operator as well as $\mathcal{B}(x;\tau,\lambda)$. It can be presented as a linear combination of the second- and first-order intertwining operators, $\mathcal{Y}(x;\tau)=-\mathcal{B}(x;\tau,\lambda)-g(\tau,\lambda)\mathcal{D}(x;\tau)$, and also may be used together with the first-order operator $\mathcal{D}(x;\tau)$ to characterize the system. At the end of this section we shall discuss the peculiarities associated with such an alternative.

[6]One could conclude that Eq. (5.4) contradicts this relation since $g(\tau,\lambda)$ diverges at $\tau=0$, and the operators $\mathcal{D}(x;\tau)$ and $\mathcal{Y}(x;\tau)$ are not defined for $\tau=0$. Equation (5.4) correctly reproduces this relation by treating $\tau=0$ as a limit $\tau\to 0$, and employing addition formulae (A6) for Jacobi elliptic functions.

[7]It can be factorized in terms of our first-order Darboux operators $\mathcal{D}$ in special cases of $\tau=(\frac{1}{2}+n)\mathbf{K}$. Such a factorization corresponds to complex values of the shift parameters, see the discussion below in this section.

Having in mind a nonperiodic limit, which we discuss later, it is convenient to fix $\lambda = \mathbf{K}/2$, and introduce the notation $\mathcal{A}(x;\tau) = \mathcal{B}(x;\tau,\frac{1}{2}\mathbf{K})$, i.e.,

$$\begin{aligned}\mathcal{A}(x;\tau) &= \mathcal{D}(x+\tfrac{1}{2}\mathbf{K};\tau+\tfrac{1}{2}\mathbf{K})\mathcal{D}^\dagger(x+\tau+\tfrac{1}{2}\mathbf{K};\tfrac{1}{2}\mathbf{K}) \\ &= -\mathcal{Y}(x;\tau) - \mathcal{C}(\tau)\mathcal{D}(x;\tau),\end{aligned} \tag{5.6}$$

where $\mathcal{C}(\tau)$ is defined in Eq. (4.8). Employing the properties of $\mathcal{Y}(x;\tau)$ and $\mathcal{D}(x;\tau)$ under Hermitian conjugation, from (5.6) one finds $\mathcal{A}^\dagger(x;\tau) = \mathcal{A}(x;-\tau)$, and then a representation alternative to (5.6) is obtained, $\mathcal{A}(x;\tau) = \mathcal{D}(x-\tau+\frac{1}{2}\mathbf{K};\frac{1}{2}\mathbf{K})\mathcal{D}^\dagger(x+\frac{1}{2}\mathbf{K};-\tau+\frac{1}{2}\mathbf{K})$. Unlike the operators $\mathcal{D}(x;\tau)$ and $\mathcal{Y}(x;\tau)$, $\mathcal{A}(x;\tau)$ is well defined at $\tau = 0$ and reduces to just a nonshifted Hamiltonian, $\mathcal{A}(x;0) = \mathcal{A}^\dagger(x;0) = H(x)$. Notice, however, that unlike $\mathcal{D}(x;\tau)$, it is not defined for $\tau = (\frac{1}{2}+n)\mathbf{K}$.

The second-order intertwining operator of the most general form (5.3) may be presented in terms of the intertwining operators $\mathcal{A}(x;\tau)$ and $\mathcal{D}(x;\tau)$, $\mathcal{B}(x;\tau,\lambda) = \mathcal{A}(x;\tau) + (C(\tau) - g(\tau,\lambda))\mathcal{D}(x;\tau)$.

Because of Eq. (5.2), the self-isospectral system possesses (for $\tau \neq (\frac{1}{2}+n)\mathbf{K}$) the second-order integrals

$$Q_1 = \begin{pmatrix} 0 & \mathcal{A}^\dagger(x;\tau) \\ \mathcal{A}(x;\tau) & 0 \end{pmatrix}, \qquad Q_2 = i\sigma_3 Q_1 \tag{5.7}$$

to be nontrivial for $\tau \neq n\mathbf{K}$ and independent from the first-order integrals (3.17).

With some algebraic manipulations, we find

$$\begin{aligned}&\mathcal{A}^\dagger(x;\tau)\mathcal{A}(x;\tau) = H(x_+)[H(x_+) + \varrho(\tau)], \\ &\quad \text{where } \varrho(\tau) = k'^2\mathrm{sn}^2 2\tau \mathrm{nc}^2 2\tau.\end{aligned} \tag{5.8}$$

A similar relation is obtained from (5.8) by a simple change $\tau \to -\tau$, $\mathcal{A}(x;\tau)\mathcal{A}^\dagger(x;\tau) = H(x_-)[H(x_-) + \varrho(\tau)]$, cf. the relations in (3.7) for the first-order intertwining operators.

The intertwining second-order operator $\mathcal{A}(x;\tau)$ annihilates the lower-energy state $\mathrm{dn}(x+\tau)$ of the system $H(x+\tau)$. Another state annihilated by it is

$$f(x,\tau) = \mathrm{dn}(x+\tau)\int^x \frac{F(u+\frac{1}{2}\mathbf{K};\tau+\frac{1}{2}\mathbf{K})}{\mathrm{dn}(u+\tau)}\,du, \tag{5.9}$$

and we have $f(x+2\mathbf{K},\tau) = \exp[2\mathbf{K}\mathrm{z}(\tau+\frac{1}{2}\mathbf{K})]f(x,\tau)$. Function (5.9) for $\tau \neq 0$ is unbounded and describes therefore a nonphysical eigenstate of $H(x+\tau)$ from the lower forbidden band with energy $E = -\varrho(\tau) < 0$, see Eq. (5.8). At $\tau = 0$, the function (5.9) reduces to $E(x+\mathbf{K})\mathrm{dn}x$, which corresponds to the nonphysical state of $H(x)$ of zero eigenvalue.

Like the first-order operator $\mathcal{D}(x;\tau)$, $\mathcal{A}(x;\tau)$ transforms the eigenstates of $H(x+\tau)$ into those of $H(x-\tau)$,

$$\mathcal{A}(x;\tau)\Psi^\alpha_\pm(x_+) = \mathcal{F}^{\mathcal{A}}_\pm(\alpha,\tau)\Psi^\alpha_\pm(x_-), \tag{5.10}$$

where

$$\begin{aligned}\mathcal{F}^{\mathcal{A}}_\pm(\alpha,\tau) &= e^{\pm i\varphi^{\mathcal{A}}(\alpha,\tau)}\mathcal{M}^{\mathcal{A}}(\alpha,\tau), \\ \mathcal{M}^{\mathcal{A}}(\alpha,\tau) &= \sqrt{E(\alpha)(E(\alpha)+\varrho(\tau))}.\end{aligned} \tag{5.11}$$

The modulus and the phase of the complex amplitude $\mathcal{F}^{\mathcal{A}}_\pm(\alpha,\tau)$ are expressed in terms of those for the first-order intertwining operator by employing Eqs. (5.1), (5.6), and (3.11),

$$\begin{aligned}\mathcal{M}^{\mathcal{A}}(\alpha,\tau) &= \mathcal{M}^{\mathcal{D}}(\alpha,\tau+\tfrac{1}{2}\mathbf{K})\mathcal{M}^{\mathcal{D}}(\alpha,\tfrac{1}{2}\mathbf{K}), \\ \varphi^{\mathcal{A}}(\alpha,\tau) &= \varphi^{\mathcal{D}}(\alpha,\tau+\tfrac{1}{2}\mathbf{K}) - \varphi^{\mathcal{D}}(\alpha,\tfrac{1}{2}\mathbf{K}).\end{aligned} \tag{5.12}$$

A phase $\varphi^{\mathcal{A}}(\alpha,\tau) \in \mathbb{R}$ has, unlike (3.13), the property $e^{i\varphi^{\mathcal{A}}(\alpha,-\tau)} = e^{-i\varphi^{\mathcal{A}}(\alpha,\tau)}$ due to the relation $\mathcal{A}^\dagger(x;\tau) = \mathcal{A}(x;-\tau)$ being different in sign from that of the first-order intertwining operator, $\mathcal{D}^\dagger(x;\tau) = -\mathcal{D}(x;-\tau)$. For the edge band states, particularly, we have $\mathcal{A}(x;\tau)\psi_i(x_+) = \mathcal{F}^{\mathcal{A}}_i(\tau)\psi_i(x_-)$, $\mathcal{A}^\dagger(x;\tau)\psi_i(x_-) = \mathcal{F}^{\mathcal{A}}_i(\tau)\psi_i(x_+)$, where $\mathcal{F}^{\mathcal{A}}_i(\tau) = 0$, $k'^2\mathrm{nc}2\tau$, $\mathrm{dn}2\tau\mathrm{nc}2\tau$, $i = 1,2,3$, cf. (3.15). The eigenstates of the integral Q_1, see (5.7), have a form similar to that for S_1,

$$\begin{aligned}Q_1\Psi^\alpha_{\pm,Q_1,\epsilon} &= \epsilon\mathcal{M}^{\mathcal{A}}(\alpha,\tau)\Psi^\alpha_{\pm,Q_1,\epsilon}, \\ \Psi^\alpha_{\pm,Q_1,\epsilon} &= \begin{pmatrix} \Psi^\alpha_\pm(x_+) \\ \epsilon e^{\pm i\varphi^{\mathcal{A}}(\alpha,\tau)}\Psi^\alpha_\pm(x_-) \end{pmatrix}, \quad \epsilon = \pm 1.\end{aligned} \tag{5.13}$$

Two relations are valid for the first and second-order intertwining operators:

$$\begin{aligned}\mathcal{D}^\dagger(x;\tau)\mathcal{A}(x;\tau) &= \mathcal{P}(x_+) - \mathcal{C}(\tau)H(x_+), \\ \mathcal{D}(x;\tau)\mathcal{A}^\dagger(x;\tau) &= -\mathcal{P}(x_-) - \mathcal{C}(\tau)H(x_-).\end{aligned} \tag{5.14}$$

Here $\mathcal{P}(x_+) = \mathcal{P}(x+\tau)$ is an anti-Hermitian third-order differential operator

$$\begin{aligned}\mathcal{P}(x_+) &= \frac{d^3}{dx^3} - \frac{3}{2}\left(\Delta^2 + \Delta' + \frac{1}{3}\delta_2(\tau)\right)\frac{d}{dx} - \frac{3}{4}(\Delta^2+\Delta')' \\ &= \frac{d^3}{dx^3} + (1 + k^2 - 3k^2\mathrm{sn}^2 x_+)\frac{d}{dx} \\ &\quad - 3k^2\mathrm{sn}x_+\mathrm{cn}x_+\mathrm{dn}x_+.\end{aligned} \tag{5.15}$$

Notice that like the Lamé Hamiltonian, the operator (5.15) is well defined for any value of the shift parameter τ. Two related equalities may be obtained from (5.14) by Hermitian conjugation.

Making use of intertwining relations (3.10) and (5.2), we find that $H(x+\tau)$ commutes with $\mathcal{D}^\dagger(x;\tau)\mathcal{A}(x;\tau)$, and, therefore, $\mathcal{P}(x+\tau)$ is an integral for the subsystem $H(x+\tau)$. For the self-isospectral supersymmetric system $\mathcal{H}$, we then have two further, third-order Hermitian integrals

$$L_1 = -i\mathrm{diag}(\mathcal{P}(x_+),\mathcal{P}(x_-)), \qquad L_2 = \sigma_3 L_1. \tag{5.16}$$

Operator $\mathcal{P}(x)$ is a Lax operator for the periodic one-gap Lamé system $H(x)$, see [38,39].

The following relations that involve the operator $\mathcal{P}(x_+)$ are valid:

$$\mathcal{D}(x;\tau)\mathcal{P}(x+\tau) = \mathcal{A}(x;\tau)[H(x_+) + \varepsilon(\tau)] + \mathcal{C}(\tau)\mathcal{D}(x;\tau)H(x_+), \tag{5.17}$$

$$\mathcal{A}(x;\tau)\mathcal{P}(x_+) = -\mathcal{D}(x;\tau)H(x_+)[H(x_+) + \varrho(\tau)] - \mathcal{C}(\tau)\mathcal{A}(x;\tau)H(x_+), \tag{5.18}$$

$$-\mathcal{P}^2(x_+) = P(H(x_+)), \qquad P(H) = H(H - k'^2)(H - 1). \tag{5.19}$$

The third-order polynomial $P(H)$ is the same spectral polynomial of the Lamé system that arose before in (2.7) and in the differential dispersion relation (2.11): it turns into zero when it acts on the edge states with energies $E_i = 0, k'^2, 1$. Since the third-order differential operator $\mathcal{P}(x_+)$ is an integral of motion for $H(x_+)$, the relation (5.19) means that the edge states $\mathrm{dn}x_+$, $\mathrm{cn}x_+$, and $\mathrm{sn}x_+$ form its kernel [39]. The spectral polynomial is a semipositive definite operator, while $\mathcal{P}(x)$ is an anti-Hermitian operator. Its action on physical Bloch states (2.3) should reduce therefore to $\pm i\sqrt{P(E(\alpha))}$. The phase cannot change abruptly within the allowed bands. To correctly fix the sign, one can consider the limit $k \to 0$, in which the Lamé system (2.1) reduces to a free particle, the integral $\mathcal{P}(x)$ reduces to a third-order operator $d^3/dx^3 + d/dx$, the forbidden zone $k'^2 < E < 1$ disappears, Bloch states transform into the plane wave states, whereas the edge states $\mathrm{dn}x$, $\mathrm{cn}x$, and $\mathrm{sn}x$ reduce, respectively, to 1, $\cos x$, and $\sin x$ with energies $E = 0$, 1, and 1. Summarizing all of this, one finds that the operator (5.15) acts on the physical Bloch states (2.3) as follows:

$$\mathcal{P}(x)\Psi^{\alpha}_{\pm}(x) = \mp i\eta(E)\sqrt{P(E(\alpha))}\Psi^{\alpha}_{\pm}(x), \tag{5.20}$$

where, as in (2.7) and (2.11), $\eta(E) = -1$ for the valence and $+1$ for the conduction bands.[8] Relation (5.20) means, particularly, that the Lax operator is not reduced just to a square root from the spectral polynomial since the Hamiltonian does not distinguish index $\pm$. This is a true, nontrivial integral of motion that is related with the Hamiltonian H by polynomial Eq. (5.19).[9] Equation (5.19) corresponds to a nondegenerate spectral elliptic curve of genus one associated with a one-gap periodic Lamé system [35].

Let us now discuss the superalgebra generated by the zero σ_3, first S_a, second Q_a, and third L_a order integrals of the motion of the self-isospectral system $\mathcal{H}$. The operator $\Gamma = \sigma_3$ commutes with L_a and anticommutes with Q_a, and so, classifies them, respectively, as bosonic and fermionic operators. Using the displayed relations for the operators $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{P}$ as well as those obtained from them by Hermitian conjugation and by the change $\tau \to -\tau$, Eq. (3.18) is extended by the anticommutation relations of the integrals S_a with Q_a, and the commutation relations of S_a and Q_a with L_a. We arrive as a result at the following superalgebra for the self-isospectral system (3.8) with the $\mathbb{Z}_2$ grading operator $\Gamma = \sigma_3$:

$$\{S_a, S_a\} = 2\delta_{ab}(\mathcal{H} + \varepsilon(\tau)), \qquad \{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}(\mathcal{H} + \varrho(\tau)), \tag{5.21}$$

$$\{S_a, Q_b\} = 2(-\delta_{ab}\mathcal{C}(\tau)\mathcal{H} + \epsilon_{ab}L_1), \tag{5.22}$$

$$[L_1, S_a] = [L_1, Q_a] = [L_1, L_2] = 0, \qquad [L_2, S_a] = 2i(S_a\mathcal{C}(\tau)\mathcal{H} + Q_a(\mathcal{H} + \varepsilon(\tau))), \tag{5.23}$$

$$[L_2, Q_a] = -2i(S_a\mathcal{H}(\mathcal{H} + \varrho(\tau)) + Q_a\mathcal{C}(\tau)\mathcal{H}), \tag{5.24}$$

$$[\sigma_3, S_a] = -2i\epsilon_{ab}S_b, \quad [\sigma_3, Q_a] = -2i\epsilon_{ab}Q_b, \quad [\sigma_3, L_a] = 0, \tag{5.25}$$

$$[\mathcal{H}, \sigma_3] = [\mathcal{H}, S_a] = [\mathcal{H}, Q_a] = [\mathcal{H}, L_a] = 0. \tag{5.26}$$

We have here a nonlinear superalgebra, in which L_1 (that is a Lax operator for $\mathcal{H}$) plays the role of the bosonic central charge, and σ_3 is treated as one of its even generators in correspondence with $\mathbb{Z}_2$ grading relations $[\sigma_3, \sigma_3] = [\sigma_3, \mathcal{H}] = [\sigma_3, L_a] = 0$ and $\{\sigma_3, S_a\} = \{\sigma_3, Q_a\} = 0$.

Since L_1 commutes with S_a and Q_a, the eigenstates (3.19) and (5.13) of S_1 and Q_1 are simultaneously the eigenstates of L_1,

$$L_1\Psi^{\alpha}_{\pm,\Lambda,\epsilon} = \mp\eta\sqrt{P(\alpha)}\Psi^{\alpha}_{\pm,\Lambda,\epsilon}, \tag{5.27}$$

where $\Lambda = S_1$ or Q_1, η is the same as in (2.11) and (5.20), and $P(\alpha) = P(E(\alpha))$. Note that unlike S_1 and Q_1, L_1 distinguishes the index $\pm$.

In correspondence with the discussion related to (5.9), the Q_a, $a = 1, 2$, annihilate the two ground states of zero energy, $\mathrm{dn}(x + \tau)$ and $\mathrm{dn}(x - \tau)$, while other two states from their kernel are nonphysical. These supercharges are not defined, however, for $\tau = (\frac{1}{2} + n)\mathbf{K}$, which are the only values of the shift parameter when the $N = 2$ supersymmetry associated with the first-order supercharges S_a is not broken. Therefore, when the first- and second-order supercharges are simultaneously defined (for $\tau \neq (\frac{1}{2} + n)\mathbf{K}, n\mathbf{K}$),

[8] Applying the first relation from (5.14) to a physical Bloch state $\Psi^{\alpha}_{+}(x_+)$ and using an equality $E(E + \varrho(\tau)) \times (E + \varepsilon(\tau)) = P(E) + \mathcal{C}^2(\tau)E^2$, we obtain the Pythagorean relation for a rectangular triangle with legs $\mathcal{C}(\tau)E(\alpha)$ and $\sqrt{P(E(\alpha))}$, $\sqrt{P(E(\alpha)) + \mathcal{C}^2(\tau)E^2(\alpha)}e^{i(\varphi^{\mathcal{D}}(\alpha,\tau+(\mathbf{K}/2)) - \varphi^{\mathcal{D}}(\alpha,\tau) - \varphi^{\mathcal{D}}(\alpha,\mathbf{K}/2))} = i\eta\sqrt{P(E(\alpha))} + \mathcal{C}(\tau)E(\alpha)$.

[9] This corresponds to Burchnall-Chaundy theorem [45] that underlies the theory of nonlinear integrable systems [35]. It asserts that if two ordinary differentials in x operators A and B of mutually prime orders l and m do commute, they obey the relation $P(A, B) = 0$, where P is a polynomial of order m in A, and of order l in B.

the supersymmetry generated together by S_a and Q_a is partially broken.

One could construct, instead, the second-order supercharges, $Q_a^{\mathcal{Y}}$, on the basis of the intertwining operators $\mathcal{Y}(x;\tau)$ and $\mathcal{Y}^\dagger(x;\tau)$. According to (5.6), they are related to Q_a as

$$Q_a^{\mathcal{Y}} = -Q_a - \mathcal{C}(\tau)S_a. \tag{5.28}$$

The corresponding superalgebra with Q_a substituted for $Q_a^{\mathcal{Y}}$ will then have a form similar to that which we have discussed, with a change in some of the corresponding (anti)-commutators for

$$\{Q_a^{\mathcal{Y}}, Q_b^{\mathcal{Y}}\} = 2\delta_{ab}(\mathcal{H}(\mathcal{H} + \varrho(\tau) - \mathcal{C}^2(\tau)) + \varepsilon(\tau)\mathcal{C}^2(\tau)), \tag{5.29}$$

$$\{S_a, Q_b^{\mathcal{Y}}\} = -2(\delta_{ab}\sigma_3\mathcal{C}(\tau)\varepsilon(\tau) + \epsilon_{ab}L_1), \tag{5.30}$$

$$[L_2, S_a] = -2i(S_a\mathcal{C}(\tau)\varepsilon(\tau) + Q_a^{\mathcal{Y}}(\mathcal{H} + \varepsilon(\tau))), \tag{5.31}$$

$$[L_2, Q_a^{\mathcal{Y}}] = 2i(S_a\mathcal{H}(\mathcal{H} + \varrho(\tau) + \varepsilon(\tau)\mathcal{C}(\tau) - \mathcal{C}^2(\tau)) + Q_a^{\mathcal{Y}}\varepsilon(\tau)\mathcal{C}(\tau)). \tag{5.32}$$

The second-order supercharges $Q_a^{\mathcal{Y}}$, like S_a, are well defined at $\tau = (\frac{1}{2} + n)\mathbf{K}$ but not defined for $\tau = n\mathbf{K}$. Analyzing the roots of the polynomial in the right-hand side of (5.29), one finds that the kernels of $Q_a^{\mathcal{Y}}$, $a = 1, 2$, for $\tau \neq (\frac{1}{2} + n)\mathbf{K}$ are formed by nonphysical states. In the exceptional case $\tau = (\frac{1}{2} + n)\mathbf{K}$, for which the supercharges Q_a are not defined, the polynomial in (5.29) reduces to the second-order polynomial

$$P_{Q^{\mathcal{Y}}}(\mathcal{H}) = (\mathcal{H} - k'^2)(\mathcal{H} - 1). \tag{5.33}$$

In correspondence with this, the zero modes of the operators $\mathcal{Y}(x;\frac{1}{2}\mathbf{K})$ and $\mathcal{Y}^\dagger(x;\frac{1}{2}\mathbf{K}) = \mathcal{Y}(x;-\frac{1}{2}\mathbf{K})$ are, respectively, the physical edge states $\mathrm{cn}(x + \frac{1}{2}\mathbf{K})$, $\mathrm{sn}(x + \frac{1}{2}\mathbf{K})$ and $\mathrm{cn}(x - \frac{1}{2}\mathbf{K})$, $\mathrm{sn}(x - \frac{1}{2}\mathbf{K})$. This property reflects a peculiarity of the case $\tau = (\frac{1}{2} + n)\mathbf{K}$ in another aspect. In accordance with footnote [5], the function $g(\tau, \lambda)$ in (5.4) turns into zero at $\lambda = \frac{1}{2}(\mathbf{K} + i\mathbf{K}')$. The second-order operator $\mathcal{Y}(x;\frac{1}{2}\mathbf{K})$ factorizes then either as $\mathcal{Y}(x;\frac{1}{2}\mathbf{K}) = -\mathcal{D}(x + \frac{1}{2}(\mathbf{K} + i\mathbf{K}'); \mathbf{K} + \frac{1}{2}i\mathbf{K}')\mathcal{D}^\dagger(x + \mathbf{K} + \frac{1}{2}i\mathbf{K}'; \frac{1}{2}(\mathbf{K} + i\mathbf{K}'))$, or in an alternative form obtained by the change of i for $-i$. These two factorizations can be presented equivalently as

$$\mathcal{Y}\left(x;\frac{1}{2}\mathbf{K}\right) = \left(\mathrm{ns}\left(x - \frac{1}{2}\mathbf{K}\right)\frac{d}{dx}\mathrm{sn}\left(x - \frac{1}{2}\mathbf{K}\right)\right) \times \left(\mathrm{cn}\left(x + \frac{1}{2}\mathbf{K}\right)\frac{d}{dx}\mathrm{nc}\left(x + \frac{1}{2}\mathbf{K}\right)\right), \tag{5.34}$$

$$\mathcal{Y}\left(x;\frac{1}{2}\mathbf{K}\right) = \left(\mathrm{nc}\left(x - \frac{1}{2}\mathbf{K}\right)\frac{d}{dx}\mathrm{cn}\left(x - \frac{1}{2}\mathbf{K}\right)\right) \times \left(\mathrm{sn}\left(x + \frac{1}{2}\mathbf{K}\right)\frac{d}{dx}\mathrm{ns}\left(x + \frac{1}{2}\mathbf{K}\right)\right). \tag{5.35}$$

From here we see that the particular case of the half-period shift of the superpartner systems is indeed exceptional. In this case not only the $N = 2$ supersymmetry associated with the first-order supercharges S_a is unbroken (when zero modes of S_a are the ground states that form a zero energy doublet), but all the other edge states of the energy doublets with $E = k'^2$ and $E = 1$ correspond to zero modes of the second-order supercharges $Q_a^{\mathcal{Y}}$. Then the third-order spectral polynomial $P(\mathcal{H}) = \mathcal{H}(\mathcal{H} - k'^2)(\mathcal{H} - 1)$ is just a product of the first- and the second-order polynomials, which correspond to the squares of the first, S_a, and the second, $Q_a^{\mathcal{Y}}$, order supercharges. In this special case the (anti-)commutation relations (5.30), (5.31), and (5.32) also simplify their form, $\{S_a, Q_b^{\mathcal{Y}}\} = -2\epsilon_{ab}L_1$, $[L_2, S_a] = -2iQ_a^{\mathcal{Y}}\mathcal{H}$, $[L_2, Q_a^{\mathcal{Y}}] = 2iS_aP_{Q^{\mathcal{Y}}}(\mathcal{H})$. We also have

$$S_aQ_a^{\mathcal{Y}} = -Q_a^{\mathcal{Y}}S_a = -iL_2, \quad S_aQ_b^{\mathcal{Y}} = Q_b^{\mathcal{Y}}S_a = -L_1, \tag{5.36}$$

where there is no summation in index a, and $b \neq a$. This is in conformity with the above-mentioned factorization of the spectral polynomial. However, since $Q_a^{\mathcal{Y}}$ does not annihilate the ground states $\mathrm{dn}(x + \frac{1}{2}\mathbf{K})$ and $\mathrm{dn}(x - \frac{1}{2}\mathbf{K})$ [which are transformed mutually by the intertwining operators $\mathcal{Y}(x;\frac{1}{2}\mathbf{K})$ and $\mathcal{Y}^\dagger(x;\frac{1}{2}\mathbf{K})$], we conclude that nonlinear supersymmetry of the self-isospectral system also is partially broken at $\tau = (\frac{1}{2} + n)\mathbf{K}$.[10]

In the next section we will see that another peculiarity of our self-isospectral system is that the choice $\Gamma = \sigma_3$ is not unique for identification of the $\mathbb{Z}_2$ grading operator: it also admits other choices for Γ, which lead to different identifications of the integrals σ_3, S_a, Q_a, and L_a as bosonic and fermionic operators. This results in alternative forms for the superalgebra. Each of such alternative forms of the superalgebra makes, particularly, a nontrivial relation (5.19) "visible" explicitly just in its structure, unlike the case with $\Gamma = \sigma_3$, which we have discussed. We also will identify the integrals of motion that detect the phases in the structure of the eigenstates of the operators S_a and Q_a.

VI. NONLOCAL $\mathbb{Z}_2$ GRADING OPERATORS

Let us introduce the operators of reflection in x and τ, $\mathcal{R}x\mathcal{R} = -x$, $\mathcal{R}\tau\mathcal{R} = \tau$, $\mathcal{R}^2 = 1$, $\mathcal{T}\tau\mathcal{T} = -\tau$, $\mathcal{T}x\mathcal{T} = x$, $\mathcal{T}^2 = 1$. They intertwine the superpartner Hamiltonians, $\mathcal{R}H(x_+) = H(x_-)\mathcal{R}$,

[10] cf. this picture as well as that for $\tau \neq (\frac{1}{2} + n)\mathbf{K}$, which we discussed above with the picture of supersymmetry breaking in the systems with topologically nontrivial Bogomolny-Prasad-Sommerfield states [46].

$\mathcal{T}H(x_+) = H(x_-)\mathcal{T}$, and we find that the self-isospectral supersymmetric system (3.8) possesses the Hermitian integrals of motion

$$\mathcal{R}\sigma_1, \quad \mathcal{T}\sigma_1, \quad \mathcal{R}\sigma_2, \quad \mathcal{T}\sigma_2, \quad \mathcal{R}\mathcal{T}\sigma_3, \quad \mathcal{R}\mathcal{T}. \tag{6.1}$$

Like for σ_3, the square of each of them equals 1. From relations

$$\begin{aligned} \mathcal{R}\mathcal{D}(x;\tau) &= \mathcal{D}^\dagger(x;\tau)\mathcal{R}, \\ \mathcal{R}\mathcal{A}(x;\tau) &= \mathcal{A}^\dagger(x;\tau)\mathcal{R}, \\ \mathcal{R}\mathcal{P}(x_+) &= -\mathcal{P}(x_-)\mathcal{R}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \mathcal{T}\mathcal{D}(x;\tau) &= -\mathcal{D}^\dagger(x;\tau)\mathcal{T}, \\ \mathcal{T}\mathcal{A}(x;\tau) &= \mathcal{A}^\dagger(x;\tau)\mathcal{T}, \\ \mathcal{T}\mathcal{P}(x_+) &= \mathcal{P}(x_-)\mathcal{T}, \end{aligned} \tag{6.3}$$

it follows that $\mathcal{R}$ and $\mathcal{T}$ also intertwine the operators of the same order within the pairs $(\mathcal{D}(x;\tau), \mathcal{D}^\dagger(x;\tau))$, $(\mathcal{A}(x;\tau), \mathcal{A}^\dagger(x;\tau))$, and $(\mathcal{P}(x_+), \mathcal{P}(x_-))$. As a result, each of nonlocal in x or τ, or in both of them, integrals of motion (6.1) either commutes or anticommutes with each of the nontrivial local integrals S_a, Q_a, and L_a. Then each integral from (6.1) also may be chosen as the $\mathbb{Z}_2$ grading operator for the self-isospectral system (3.8). The corresponding $\mathbb{Z}_2$ parities, together with those prescribed by a local integral σ_3, are shown in Table I. The $\mathbb{Z}_2$ parities of the second-order integrals $Q_a^{\mathcal{Y}}$, defined in (5.28), are also displayed; the equality $\mathcal{C}(-\tau) = -\mathcal{C}(\tau)$ has to be employed in their computation. Notice that $Q_a^{\mathcal{Y}}$, $a = 1, 2$ always has the same $\mathbb{Z}_2$ parity as the Q_a with the same value of the index a.

A positive $\mathbb{Z}_2$ parity is assigned for the Hamiltonian $\mathcal{H}$ by any of the integrals (6.1). Then for any choice of the grading operator presented in Table I, four of the eight local integrals σ_3, $\mathcal{H}$, S_a, L_a, and Q_a or $Q_a^{\mathcal{Y}}$ are identified as bosonic generators, and four are identified as fermionic generators of the corresponding nonlinear superalgebra. The superalgebra may be found for each choice of Γ from the set of integrals (6.1) by employing the quadratic products of the operators $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{P}$, which have been discussed in the previous section. Alternatively, some of the (anti)-commutators may be obtained with the help of the already known (anti)-commutation relations and relations between the generators that involve σ_3. For instance, $[S_1, Q_1] = i\sigma_3\{S_1, Q_2\}$. As an example, we display the explicit form of the superalgebraic relations for the choice $\Gamma = \mathcal{R}\mathcal{T}$,

$$\begin{aligned} \{S_a, S_b\} &= 2\delta_{ab}(\mathcal{H} + \varepsilon(\tau)), \\ \{S_a, L_1\} &= 2\epsilon_{ab}(Q_b(\mathcal{H} + \varepsilon(\tau)) + \mathcal{C}(\tau)S_b\mathcal{H}), \\ \{S_a, L_2\} &= 0, \end{aligned} \tag{6.4}$$

$$\begin{aligned} \{L_1, L_1\} &= \{L_2, L_2\} = 2P(\mathcal{H}), \\ \{L_1, L_2\} &= 2\sigma_3 P(\mathcal{H}), \end{aligned} \tag{6.5}$$

$$\begin{aligned} [Q_a, S_b] &= 2i(-\delta_{ab}L_2 + \epsilon_{ab}\mathcal{C}(\tau)\sigma_3\mathcal{H}), \\ [Q_1, Q_2] &= -2i\sigma_3\mathcal{H}(\mathcal{H} + \varrho(\tau)), \end{aligned} \tag{6.6}$$

$$\begin{aligned} [Q_a, L_1] &= 0, \\ [Q_a, L_2] &= 2i(\mathcal{C}(\tau)Q_a\mathcal{H} + S_a\mathcal{H}(\mathcal{H} + \varrho(\tau))), \end{aligned} \tag{6.7}$$

which should be supplied by the commutation relations (5.25) and (5.26). $P(\mathcal{H})$ in (6.5) is the spectral polynomial, see (5.19).

A fundamental polynomial relation (5.19) between the Lax operator and the Hamiltonian, that underlies a very special, finite-gap nature of the Lamé system,[11] does not show up in the superalgebraic structure for the usual choice of the diagonal matrix σ_3 as the grading operator Γ, but is involved explicitly in the superalgebra in the form of the anticommutator of one or both generators L_a, $a = 1, 2$, when any of six nonlocal integrals (6.1) are identified as Γ.

Note that for $\Gamma = \mathcal{R}\mathcal{T}$ as well as for any other choice of the grading operator that involves the operator $\mathcal{T}$, the constant $\mathcal{C}(\tau)$ anticommutes with the grading operator and should be treated as an odd generator of the superalgebra. As a result, the right-hand side in the second anticommutator in (6.4) is an even operator, while the right-hand side in the first (second) commutator in (6.6) [in (6.7)] is an odd operator, as it should be.

By employing Eq. (5.28), one can rewrite the superalgebraic relations (6.4), (6.6), and (6.7) in terms of the integrals $Q_a^{\mathcal{Y}}$, which, unlike Q_a, are defined for $\tau = (\frac{1}{2} + n)\mathbf{K}$. We do not display them here, but write down only a commutation relation

$$[S_a, Q_b^{\mathcal{Y}}] = 2i(\delta_{ab}L_2 + \sigma_3\epsilon_{ab}\mathcal{C}(\tau)\varepsilon(\tau)), \tag{6.8}$$

which we will need below. The form of such a superalgebra simplifies significantly at $\tau = (\frac{1}{2} + n)\mathbf{K}$ in correspondence with the special nature that the integrals S_a and $Q_a^{\mathcal{Y}}$ acquire in that case. Particularly, one finds

TABLE I. $\mathbb{Z}_2$ parities of the local integrals.

Γ	σ_3	S_1	S_2	$Q_1, Q_1^{\mathcal{Y}}$	$Q_2, Q_2^{\mathcal{Y}}$	L_1	L_2
σ_3	+	−	−	−	−	+	+
$\mathcal{R}\sigma_1$	−	+	−	+	−	−	+
$\mathcal{T}\sigma_1$	−	−	+	+	−	+	−
$\mathcal{R}\sigma_2$	−	−	+	−	+	−	+
$\mathcal{T}\sigma_2$	−	+	−	−	+	+	−
$\mathcal{R}\mathcal{T}\sigma_3$	+	+	+	−	−	−	−
$\mathcal{R}\mathcal{T}$	+	−	−	+	+	−	−

[11] In a generic situation the spectrum of a one-dimensional periodic system has infinitely many gaps [35].

$$\{S_a, S_b\} = 2\delta_{ab}\mathcal{H}, \qquad \{S_a, L_1\} = -2\epsilon_{ab}Q_b^{\mathcal{Y}}\mathcal{H}, \quad (6.9)$$

$$[Q_a^{\mathcal{Y}}, S_b] = 2i\delta_{ab}L_2, \qquad [Q_1^{\mathcal{Y}}, Q_2^{\mathcal{Y}}] = -2i\sigma_3 P_{Q^{\mathcal{Y}}}(\mathcal{H}),$$
$$[L_2, Q_a^{\mathcal{Y}}] = 2iS_a P_{Q^{\mathcal{Y}}}(\mathcal{H}). \quad (6.10)$$

All the integrals (6.1), including σ_3 but excluding $\mathcal{RT}$, may be related between themselves by unitary transformations, whose generators are constructed in terms of the grading operators themselves. For instance, $U\sigma_3 U^\dagger = \mathcal{R}\sigma_1 = \tilde{\sigma}_3$, $U = U^\dagger = U^{-1} = \frac{1}{\sqrt{2}}(\sigma_3 + \mathcal{R}\sigma_1)$. Being constructed from the integrals of motion, such a transformation does not change the supersymmetric Hamiltonian $\mathcal{H}$. On the other hand, if we apply it to any nontrivial integral, the transformed operator will still be an integral. Particularly, its application to the integrals S_1 and Q_1 gives

$$\tilde{S} = i\mathcal{R}\sigma_2 S_1 = \mathrm{diag}(\mathcal{R}\mathcal{D}(x;\tau), -\mathcal{R}\mathcal{D}^\dagger(x;\tau)),$$
$$\tilde{Q} = i\mathcal{R}\sigma_2 Q_1 = \mathrm{diag}(\mathcal{R}\mathcal{A}(x;\tau), -\mathcal{R}\mathcal{A}^\dagger(x;\tau)). \quad (6.11)$$

These are nontrivial Hermitian *nonlocal* integrals of motion for the self-isospectral system (3.8).[12] Equation (6.11) has a sense of Foldy-Wouthuysen transformation that diagonalizes the supercharges S_1 and Q_1. The price we pay for this is the nonlocality of the transformed operators.

Multiplication of (6.11) by the grading operators gives further nonlocal integrals, particularly, $\sigma_3\tilde{S}$ and $\sigma_3\tilde{Q}$. Since both operators (6.11) are diagonal, the Lamé subsystem $H(x_+)$ may be characterized, in addition to the Lax integral $\mathcal{P}(x_+)$, by two nontrivial nonlocal integrals:

$$\hat{S} = \mathcal{R}\mathcal{D}(x;\tau), \qquad \hat{Q} = \mathcal{R}\mathcal{A}(x;\tau). \quad (6.12)$$

In correspondence with relations $\mathcal{D}^\dagger(x;\tau) = -\mathcal{D}(x;-\tau)$ and $\mathcal{A}^\dagger(x;\tau) = \mathcal{A}(x;-\tau)$, another subsystem $H(x_-)$ is then characterized by integrals of the same form but with τ changed to $-\tau$. The operator $\hat{\Gamma} = \mathcal{RT}$ is an integral for the subsystem $H(x_+)$ [as well as for subsystem $H(x_-)$]. It can be identified as a $\mathbb{Z}_2$ grading operator that assigns definite $\mathbb{Z}_2$ parities for the nontrivial integrals of the subsystem $H(x_+)$. Namely, in correspondence with (6.2) and (6.3), the integrals $-i\mathcal{P}(x_+)$ and $\hat{S}$ are fermionic operators with respect to such a grading, while $\hat{Q}$ should be treated as a bosonic operator. Multiplying the fermionic integrals by $i\hat{\Gamma}$ and the bosonic integral by $\hat{\Gamma}$, we obtain three more integrals for $H(x_+)$. It is not difficult to calculate the corresponding superalgebra generated by these integrals. Let us note only that since the described supersymmetry may be revealed in the subsystem $H(x_+)$ [or, in $H(x_-)$], it may be treated as a bosonized supersymmetry, see [37,38,47].

[12]Notice that the (1 + 1)-dimensional GN model has a system of infinitely many (nonlocal) conservation laws.

Let us return to the question of degeneration in our self-isospectral system. This will allow us to observe some other interesting properties related to the nonlocal integrals (6.1). Let us take a pair of mutually commuting integrals S_1 and L_1. They can be simultaneously diagonalized, and for their common eigenstates we have $S_1\Psi^\alpha_{\pm,S_1,\epsilon} = \epsilon\mathcal{M}^{\mathcal{D}}(\alpha,\tau)\Psi^\alpha_{\pm,S_1,\epsilon}$ and $L_1\Psi^\alpha_{\pm,S_1,\epsilon} = \mp\eta(\alpha)\sqrt{P(\alpha)}\Psi^\alpha_{\pm,S_1,\epsilon}$, see Eqs. (3.19) and (5.27). We can distinguish all four states by these relations for any value of the energy within the valence and conduction bands, and each two doublet states for the edges $E = 0$, k'^2, 1 of the allowed bands when $\tau \neq (\frac{1}{2} + n)\mathbf{K}$. However, in the case of $\tau = (\frac{1}{2} + n)\mathbf{K}$, the two ground states of zero energy are annihilated by both operators S_1 and L_1, and cannot be distinguished by them. In this special case the operator σ_3 commutes with S_1 and L_1 on the subspace $E = 0$, and may be used to distinguish the two ground states. It is necessary to remember, however, that σ_3 does not commute with S_1 on the subspaces of nonzero energy.

There is yet another possibility. According to Table I, the local integrals S_1 and L_1 commute with the nonlocal integral $\mathcal{T}\sigma_2$. We then find

$$\mathcal{T}\sigma_2\Psi^\alpha_{\pm,S_1,\epsilon} = i\epsilon e^{\mp i\varphi^{\mathcal{D}}(\alpha,\tau)}\Psi^\alpha_{\pm,S_1,\epsilon}, \quad (6.13)$$

where we used relation (3.14). The operator $\mathcal{T}\sigma_2$ therefore detects the phase in the structure of the eigenstates of S_1. By comparing the two supersymmetric systems with the shift parameters τ and $\tau + \mathbf{K}$, and by taking into account the $2\mathbf{K}$ periodicity of the Θ function in (3.12) and the $2\mathbf{K}$ antiperiodicity of sn$u$, we get from (3.14) that $e^{i(\varphi^{\mathcal{D}}(\alpha,\tau+\mathbf{K})-\varphi^{\mathcal{D}}(\alpha,\tau))} = e^{(i/\mathbf{K})\kappa(\alpha)\tau}$. Hence, the integral $\mathcal{T}\sigma_2$ does the same job as the *translation for the period* operator (which is also a nonlocal integral for the system): it allows us to determine an energy-dependent quasimomentum. Finally, in the case of zero energy ($\alpha = \mathbf{K} + i\mathbf{K}'$), treating $\tau = (\frac{1}{2} + n)\mathbf{K}$ as a limiting case, one can also distinguish two ground states in the supersymmetric doublet by means of (6.13).

Instead of S_1, L_1, and $\mathcal{T}\sigma_2$, we could choose the triplet S_2, L_1, and $\mathcal{T}\sigma_1$ of mutually commuting integrals, see Table I. The states within the supermultiplets can also be distinguished by choosing the triplets of mutually commuting integrals $(Q_1, L_1, \mathcal{T}\sigma_1)$, or $(Q_2, L_1, \mathcal{T}\sigma_2)$. For the two latter cases, the doublet of the ground states is annihilated by Q_a and L_1 for any value of the shift parameter τ (excluding the case $\tau = (\frac{1}{2} + n)\mathbf{K}$ when Q_a are not defined), but the corresponding integrals $\mathcal{T}\sigma_1$ or $\mathcal{T}\sigma_2$ do the necessary job of distinguishing the states as well.

The integrals $\mathcal{R}\sigma_1$ and $\mathcal{RT}\sigma_3$ act on the eigenstates of S_1, with which they also commute, as $\mathcal{R}\sigma_1\Psi^\alpha_{\pm,S_1,\epsilon}(x,\tau) = -\epsilon e^{\pm i\varphi^{\mathcal{D}}(\alpha,\tau)}\Psi^\alpha_{\mp,S_1,\epsilon}(x,\tau)$, $\mathcal{RT}\sigma_3\Psi^\alpha_{\pm,S_1,\epsilon}(x,\tau) = -\Psi^\alpha_{\mp,S_1,\epsilon}(x,\tau)$. These operators interchange the states with the + and − indexes, and anticommute with the integral L_1. The edge states,

which do not carry such an index, are annihilated by L_1, so that there is no contradiction with the information presented in Table I.

In concluding this section, we note that the Witten index computed with the grading operator identified with any of the six nonlocal integrals (6.1) is the same as for a choice $\Gamma = \sigma_3$, i.e., $\Delta_W = 0$.

VII. SUPERSYMMETRY OF THE ASSOCIATED PERIODIC BDG SYSTEM

Until now, we have discussed the self-isospectrality of the one-gap Lamé system with the second-order Hamiltonian. Though we have shown that its supersymmetric structure is much richer than the usual one, from the viewpoint of the physics of the GN model, it is more natural to look at the revealed picture from another perspective.

Let us take one of the first-order integrals S_a of the self-isospectral Lamé system, say S_1, and consider it as a first-order Dirac Hamiltonian. In such a way we obtain an intimately related, but different physical system. Unlike the second-order operator $\mathcal{H}$, the spectrum (3.19) of S_1 depends on τ. We get a periodic Bogoliubov-de Gennes system with the Hamiltonian $H_{\mathrm{BdG}} = S_1$. The interpretation of the function $\Delta(x;\tau)$ changes in this case: this is the Dirac scalar potential in correspondence with the discussion from Sec. I. With a dependence on a physical context, it takes a sense of an order parameter, a condensate, or a gap function.

The τ-dependent spectrum of such a BdG system consists of four or three allowed bands located symmetrically with respect to the level $\mathcal{E} = 0$, see Fig. 3. The interpretation of the bands also changes and depends on the physical

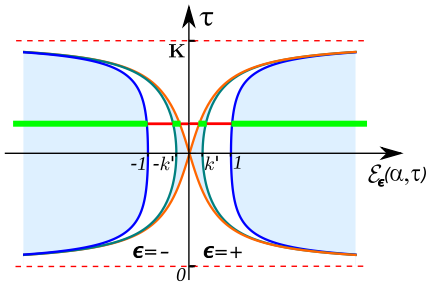

FIG. 3 (color online). The spectrum of $H_{\mathrm{BdG}} = S_1$ possesses symmetries $\mathcal{E}_\epsilon(\alpha,\tau) = \mathcal{E}_\epsilon(\alpha,-\tau) = \mathcal{E}_\epsilon(\alpha,\tau+\mathbf{K})$, $\mathcal{E}_\epsilon(\alpha,\frac{1}{2}\mathbf{K}+\tau) = \mathcal{E}_\epsilon(\alpha,\frac{1}{2}\mathbf{K}-\tau)$, and $\mathcal{E}_-(\alpha,\tau) = -\mathcal{E}_+(\alpha,\tau)$. The horizontal line shows a spectrum for some value of τ, $\frac{1}{2}\mathbf{K} < \tau < \mathbf{K}$. The allowed (forbidden) bands on it are presented by thick green (thin red) intervals, whose points are distinguished by the parameter α, see Eq. (7.2). Curves indicate the edges of the allowed bands (7.1). The point $\mathcal{E}_\epsilon(\mathbf{K}+i\mathbf{K}',\frac{1}{2}\mathbf{K}) = 0$ corresponds to a doubly degenerate energy level in the allowed band $(-k', k')$, that is formed by the two merging at the $\tau = \frac{1}{2}\mathbf{K}$ internal allowed bands.

context. For $\tau \neq (\frac{1}{2}+n)\mathbf{K}$, the positive and negative "internal" bands are separated by a nonzero gap $\Delta\mathcal{E}(\tau) = 2\sqrt{\varepsilon(\tau)} = 2|\mathrm{cn}2\tau\mathrm{ns}2\tau|$, which disappears at $\tau = (\frac{1}{2}+n)\mathbf{K}$. The total number of gaps in the spectrum is three in the case $\tau \neq (\frac{1}{2}+n)\mathbf{K}$, $\mathcal{E} \in (-\infty,\mathcal{E}_{3,-}]\cup[\mathcal{E}_{2,-},\mathcal{E}_{1,-}]\cup[\mathcal{E}_{1,+},\mathcal{E}_{2,+}]\cup[\mathcal{E}_{3,+},\infty)$, while for $\tau = (\frac{1}{2}+n)\mathbf{K}$ there are only two gaps, $\mathcal{E} \in (-\infty,\mathcal{E}_{3,-}]\cup[\mathcal{E}_{2,-},\mathcal{E}_{2,+}]\cup[\mathcal{E}_{3,+},\infty)$. According to (3.15), (3.16), and (3.19), the edges $\mathcal{E}_{i,\epsilon}$ of the internal ($i = 1, 2$) and external ($i = 3$) allowed bands are

$$\mathcal{E}_{1,\epsilon}(\tau) = \epsilon\sqrt{\varepsilon(\tau)}, \qquad \mathcal{E}_{2,\epsilon}(\tau) = \epsilon\sqrt{k'^2+\varepsilon(\tau)}, \qquad \mathcal{E}_{3,\epsilon}(\tau) = \epsilon\sqrt{1+\varepsilon(\tau)}, \tag{7.1}$$

where $\epsilon = \pm$, and the eigenstates have the form $\Psi_{i,\epsilon}(x;\tau) = (\psi_i(x_+), \epsilon e^{i\varphi_i^{\mathcal{D}}(\tau)}\psi_i(x_-))^T$, $S_1\Psi_{i,\epsilon}(x;\tau) = \mathcal{E}_{i,\epsilon}\Psi_{i,\epsilon}(x;\tau)$.

In the context of the physics of conducting polymers, for example, the internal bands are referred to as the lower, $[\mathcal{E}_{2,-},\mathcal{E}_{1,-}]$, and upper, $[\mathcal{E}_{1,+},\mathcal{E}_{2,+}]$, polaron bands; the upper external band, $[\mathcal{E}_{3,+},\infty)$, is called the conduction band; the lower external band, $(-\infty,\mathcal{E}_{3,-}]$, is referred to as the valence band [31]. In the general case for eigenstates (3.19), we have

$$S_1\Psi^{\alpha}_{\pm,S_1,\epsilon}(x;\tau) = \mathcal{E}_\epsilon(\alpha,\tau)\Psi^{\alpha}_{\pm,S_1,\epsilon}(x;\tau), \qquad \mathcal{E}_\epsilon(\alpha,\tau) = \epsilon\sqrt{E(\alpha)+\varepsilon(\tau)}, \tag{7.2}$$

where $E(\alpha)$ for internal and external bands is given by Eqs. (2.5) and (2.6).

Since $H_{\mathrm{BdG}} = S_1$ does not distinguish the index $\pm$ of the wave functions within the allowed bands, each corresponding energy level is doubly degenerate. Six edge states for $\tau \neq (\frac{1}{2}+n)\mathbf{K}$ are singlets. In the case of $\tau = (\frac{1}{2}+n)\mathbf{K}$, four edge states with energies $\mathcal{E} = \pm k'$ and ± 1 are singlets. Zero energy states $\Psi_{1,\epsilon}$ form a doublet in this case, as happens for any other energy level inside any allowed band.

The described degeneration in the spectrum of S_1 indicates that the BdG system might possess its *own nonlinear supersymmetric structure*. This is indeed the case. First, from Table I we see that there are three operators, $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$, and $\mathcal{R}\mathcal{T}\sigma_3$, which commute with S_1, and the square of each equals one. Hence, each of them may be chosen as a $\mathbb{Z}_2$ grading operator for the BdG system. There are three more, nontrivial local integrals of motion for H_{BdG}. One is the second-order operator $\mathcal{H}$. This, however, is not interesting from the viewpoint of a supersymmetric structure since it is just a shifted square of $H_{\mathrm{BdG}} = S_1$, $\mathcal{H} = S_1^2 - \varepsilon(\tau)$. Then we have a third-order integral $L_1 \equiv \mathcal{L}_1$, which has been identified before as the Lax operator for the self-isospectral Lamé system $\mathcal{H}$. Finally, the fourth-order operator $\mathcal{G}_1 = S_1\mathcal{L}_1$ is also identified as a local integral of motion. Note that both integrals $\mathcal{L}_1$ and $\mathcal{G}_1$ distinguish the states inside the allowed bands,

which differ in the index $\pm$. On distinguishing the states with $\mathcal{E}=0$ to be present in the spectrum if $\tau=(\frac{1}{2}+n)\mathbf{K}$, see the discussion at the end of the previous section. Further nontrivial but nonlocal integrals may be obtained if we multiply the local integrals by the operators $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$, and $\mathcal{R}\mathcal{T}\sigma_3$. Then, as in the case of the self-isospectral Lamé system, different choices for the grading operator lead to distinct identifications of the $\mathbb{Z}_2$ parities of the integrals.

For the sake of definiteness, let us choose $\Gamma=\mathcal{R}\sigma_1$, and assume first that $\tau\neq(\frac{1}{2}+n)\mathbf{K}$. The other two possibilities for the choice of Γ may be considered in an analogous way. If, additionally, we restrict our analysis by the integrals that do not include in their structure nonlocal in τ operator $\mathcal{T}$, we get two $\mathbb{Z}_2$-even (commuting with Γ) integrals in addition to $H_{\mathrm{BdG}}=S_1$, namely, $\mathcal{R}\sigma_1$ and $\mathcal{R}\sigma_1 S_1$. The four $\mathbb{Z}_2$-odd (anticommuting with Γ) integrals are $\mathcal{L}_1$, $\mathcal{G}_1$, $\mathcal{L}_2=i\mathcal{R}\sigma_1\mathcal{L}_1$, and $\mathcal{G}_2=i\mathcal{R}\sigma_1\mathcal{G}_1$. All of these integrals are Hermitian operators. It is interesting to note that a nonlocal integral $\mathcal{R}\sigma_1 S_1$ is related to one of the diagonal nonlocal operators from (6.11), $\mathcal{R}\sigma_1 S_1=\sigma_3\tilde{S}$. A nonlocal diagonal operator $\mathcal{G}_2$ may also be related to (6.11), $\mathcal{G}_2=\tilde{Q}S_1^2+\mathcal{C}(\tau)\tilde{S}(S_1^2-\varepsilon(\tau))$. Since, however, integrals $\mathcal{R}\sigma_1 S_1$ and $\mathcal{G}_a$ are just the integrals $\mathcal{R}\sigma_1$ and $\mathcal{L}_a$ multiplied by the BdG Hamiltonian S_1, we can omit them as well as $\mathcal{H}$. We then obtain the nontrivial (anti)commutation relations of the nonlinear BdG superalgebra,

$$[\mathcal{R}\sigma_1,\mathcal{L}_a]=-2i\epsilon_{ab}\mathcal{L}_b,\quad \{\mathcal{L}_a,\mathcal{L}_b\}=2\delta_{ab}\hat{P}(S_1,\tau). \tag{7.3}$$

Here, in correspondence with Eqs. (5.19), (5.21), and (6.5), $\hat{P}(S_1,\tau)$ is the sixth-order spectral polynomial of the BdG system,

$$\hat{P}(S_1,\tau)=(S_1^2-\varepsilon(\tau))(S_1^2-\varepsilon(\tau)-k'^2)(S_1^2-\varepsilon(\tau)-1), \tag{7.4}$$

whose six roots correspond to the energy levels (7.1).

Superalgebra (7.3) has a structure similar to that of a hidden, bosonized supersymmetry [47] of the unextended Lamé system (2.1), which was revealed in [38]. There, the role of the grading operator is played by a reflection operator $\mathcal{R}$, the matrix integrals $\mathcal{L}_a$ are substituted by the Lax operator $-i\mathcal{P}(x)$, see Eq. (5.15), and by $\mathcal{R}\mathcal{P}(x)$. The sixth-order polynomial $\hat{P}(S_1,\tau)$ of the BdG Hamiltonian S_1 is changed there for a third-order spectral polynomial $P(H)$, see Eq. (5.19).

We have seen that the structure of the BdG spectrum changes significantly at $\tau=(\frac{1}{2}+n)\mathbf{K}$. Essential changes also happen in the superalgebraic structure. Indeed, from (6.8) it follows that $[S_1,Q_2^{\mathcal{Y}}]=2i\sigma_3\epsilon_{ab}\mathcal{C}(\tau)\varepsilon(\tau)$, i.e., in a generic case $Q_2^{\mathcal{Y}}$ does not commute with H_{BdG}. Contrarily, for $\tau=(\frac{1}{2}+n)\mathbf{K}$ this is an additional nontrivial, second-order integral of motion of the BdG system. This integral, like the third-order integral L_1, also distinguishes the states marked by the index $\pm$ inside the allowed bands, $Q_2^{\mathcal{Y}}\Psi^{\alpha}_{\pm,S_1,\epsilon}=\pm\eta\sqrt{P_{Q^{\mathcal{Y}}}(E(\alpha))}\Psi^{\alpha}_{\pm,S_1,\epsilon}$, where η is the same as in (2.11) and (5.20), i.e., $\eta=-1$ for $0\leq E\leq k'^2$ and $\eta=+1$ for $E\geq 1$, while $P_{Q^{\mathcal{Y}}}(E)$ is a polynomial that appeared earlier in (5.33), i.e., $P_{Q^{\mathcal{Y}}}(E)=(E-k'^2)(E-1)$. In this case, L_1 is no longer an independent integral for the BdG system, since here $L_1=-S_1Q_2^{\mathcal{Y}}$ in correspondence with (5.36). The integral $Q_2^{\mathcal{Y}}$ anticommutes with $\mathcal{R}\sigma_1$ and $\mathcal{R}\mathcal{T}\sigma_3$. Let us choose, again, $\Gamma=\mathcal{R}\sigma_1$, and denote $\mathcal{Q}_1=Q_2^{\mathcal{Y}}$ and $\mathcal{Q}_2=i\Gamma\mathcal{Q}_1$. Instead of (7.3), we get a nonlinear superalgebra of the order four,

$$[\mathcal{R}\sigma_1,\mathcal{Q}_a]=-2i\epsilon_{ab}\mathcal{Q}_b,\quad \{\mathcal{Q}_a,\mathcal{Q}_b\}=2\delta_{ab}\hat{P}_{\mathcal{Q}}(S_1), \tag{7.5}$$

where $\hat{P}_{\mathcal{Q}}(S_1)=(S_1^2-k'^2)(S_1^2-1)$.

It is interesting to see what happens with the Witten index in the described unusual supersymmetry of the BdG system with the first-order Hamiltonian. One can construct the eigenstates of the grading operator $\Gamma=\mathcal{R}\sigma_1$,

$$\begin{aligned}&\Gamma\Psi^{(\epsilon)}(x;\alpha,\tau)=-\epsilon\Psi^{(\epsilon)}(x;\alpha,\tau),\\&\Psi^{(\epsilon)}(x;\alpha,\tau)\equiv\Psi^{\alpha}_{+,S_1,\epsilon}(x;\tau)+e^{i\varphi^{\mathcal{D}}(\alpha,\tau)}\Psi^{\alpha}_{-,S_1,\epsilon}(x;\tau).\end{aligned} \tag{7.6}$$

For any energy value inside any allowed band [including $\mathcal{E}=0$ in the case of $\tau=(\frac{1}{2}+n)\mathbf{K}$], we have two states with opposite eigenvalues of Γ, and these contribute zero into the Witten index $\Delta_W=\mathrm{Tr}\Gamma$, where the trace is taken over all the eigenstates of the grading operator Γ. On the other hand, the edge states $\Psi_{i,\epsilon}(x,\tau)$ are singlets. They are also the eigenstates of Γ. The eigenstates of opposite energy signs have opposite eigenvalues, $+1$ and -1, of the grading operator. As a result, we conclude that the Witten index Δ_W in such a supersymmetric system equals zero for any value of τ [i.e., for $\tau\neq(\frac{1}{2}+n)\mathbf{K}$ when there are no zero energy states in the spectrum, and for $\tau=(\frac{1}{2}+n)\mathbf{K}$ when the spectrum contains a doublet of zero energy states], like this happens in the self-isospectral Lamé system with the second-order supersymmetric Hamiltonian. The same result $\Delta_W=0$ is obtained for the choices $\Gamma=\mathcal{T}\sigma_2$ and $\Gamma=\mathcal{R}\mathcal{T}\sigma_3$.

Finally, it is worth noting that in accordance with the structure of superalgebra (7.3), the third-order matrix BdG supercharges $\mathcal{L}_a$ annihilate all the six edge eigenstates of $H_{\mathrm{BdG}}=S_1$ in the case of $\tau\neq(\frac{1}{2}+n)\mathbf{K}$. In the special cases $\tau=(\frac{1}{2}+n)\mathbf{K}$ a central gap disappears in the spectrum, and, consistently with (7.5), all the remaining four edge states are the zero modes of the second-order matrix BdG supercharges $\mathcal{Q}_a$. In other words, the spectral changes that happen in the BdG system at special values of the parameter $\tau=(\frac{1}{2}+n)\mathbf{K}$, which correspond to a zero value of the bare mass m_0 in the GN model (1.2), are reflected coherently by the changes in its superalgebraic structure.

VIII. INFINITE PERIOD LIMIT

Let us now discuss the infinite period limit of our self-isospectral Lamé and the associated BdG systems, i.e., the case when the period $2\mathbf{K}$ tends to infinity.

$\mathbf{K} \to \infty$ assumes [13] $k \to 1$, $k' \to 0$, $\mathbf{K}' \to \frac{1}{2}\pi$, and relations (A5) and (B8) have to be employed. According to (B8) and (B9), a limit for the quotient of Θ functions is also well defined,

$$\lim_{k\to 1} \frac{\Theta(u)}{\Theta(v)} = \frac{\cosh(u)}{\cosh(v)}, \qquad u, v \in \mathbb{C}. \tag{8.1}$$

The periodic Lamé Hamiltonian (2.1) transforms in this limit into a reflectionless one-gap Pöschl-Teller Hamiltonian

$$H_{\mathrm{PT}}(x) = -\frac{d^2}{dx^2} - \frac{2}{\cosh^2 x} + 1. \tag{8.2}$$

When the limit $\mathbf{K} \to \infty$ is applied to the self-isospectral system (3.8), we assume that a shift parameter τ remains to be finite. As a result, we get a self-isospectral nonperiodic PT system,

$$\mathcal{H}_{\mathrm{PT}}(x) = \mathrm{diag}(H_\tau(x), H_{-\tau}(x)), \tag{8.3}$$

where $H_\tau(x) = H_{\mathrm{PT}}(x+\tau)$ and $H_{-\tau}(x) = H_{\mathrm{PT}}(x-\tau)$. In what follows we trace out how the peculiar supersymmetry of the self-isospectral Lamé system transforms in the infinite period limit into the supersymmetric structure of the system (8.3), which was studied recently in [40].

Since the superpartners in (8.3) are the two mutually shifted copies of the same PT system, it is clear that the limit does not change the Witten index: it remains to be equal to zero as in the periodic case. In general, however, the index may or may not change depending on the concrete form of the self-isospectral Lamé system to which the limit is applied. For instance, in the case of the system with superpartners $H(x)$ and $H(x+\mathbf{K})$ [see the remark just below Eq. (3.7)], the infinite period limit gives, instead of (8.3), a supersymmetric system with one superpartner to be the PT system (8.2), while another one [which is a limit of $H(x+\mathbf{K})$] to be a free particle $H_0 = -\frac{d^2}{dx^2} + 1$. Superpartner potentials in such a supersymmetric (but not self-isospectral) system are distinct. The only difference in the spectrum for the system (8.2) from that of H_0 consists in the presence of a unique bound state, see below. Consequently, the Witten index changes in the infinite period limit, by taking a value of the modulus one. If in the system (3.8) one takes $\tau = \tau(\mathbf{K})$ such that $\tau \to \infty$ for $\mathbf{K} \to \infty$, the limit then produces a trivial self-isospectral system composed from the two copies of the free particle Hamiltonian H_0. In such a case, the Witten index does not change in agreement with (8.3) and (8.2).

[13] Any of these four limits assumes three others.

The listed examples also mean that the shifts for the period, in a sense, "interfere" with the infinite period limit. The self-isospectral Lamé system composed from $H(x_+)$ and $H(x_-)$ is equivalent, for instance, to a system with superpartner Hamiltonians $H(x_+)$ and $H(x_- + 2\mathbf{K})$.[14] If before taking a limit we do not "eliminate" the period $2\mathbf{K}$ shift in the second subsystem, we will obtain a (not self-isospectral) system with superpartners H_τ and H_0 instead of (8.3).

Let us return to the symmetric case of the self-isospectral Lamé system (3.8), whose infinite period limit corresponds to the self-isospectral PT system (8.3). All the energy values (2.5) of the valence band transform into zero in the infinite period limit because of $k' \to 0$, i.e., this entire band shrinks into one energy level $E = 0$ for the system (8.2). In conformity with this, all of the Bloch states (2.3) of this band, including the edge states $\mathrm{dn}x$ and $\mathrm{cn}x$, turn into the unique bound state $\frac{1}{\cosh x}$ of $E = 0$ for the PT system.[15] Then the states $1/\cosh(x \pm \tau)$ form a supersymmetric doublet of the ground states for the self-isospectral system (8.3). The doublet of the edge states $\mathrm{sn}(x \pm \tau)$ of the system (3.8) transforms into a doublet of the lowest states $\tanh(x \pm \tau)$ of the energy $E = 1$ in the scattering sector of the spectrum for (8.3). It is interesting to see how the eigenstates with $E > 1$ in the scattering sector of the PT system originate from the Bloch states (2.3). The energy (2.6) as a function of the parameter β, which in the infinite period limit takes values in the interval $0 \le \beta < \frac{\pi}{2}$, reduces to $E(i\beta) = \frac{1}{\cos^2\beta} \ge 1$. The states (2.3) transform into $\Psi_\pm^{i\beta}(x) = \cos\beta(\tanh x \pm i\tan\beta)\exp(\mp i x\tan\beta)$. Denoting $\tan\beta = \mathbb{k} \ge 0$, we obtain $E = 1 + \mathbb{k}^2$, and the states $\Psi_\pm^{i\beta}(x)$ take the form of the scattering eigenstates of the PT system, $\Psi_\pm^{i\beta}(x) \to \Psi^{\pm\mathbb{k}}(x) = -\frac{1}{\sqrt{E}}(\pm i\mathbb{k} - \tanh x)e^{\pm i\mathbb{k}x}$.

We have

$$F(x;\tau) \overset{k\to 1}{\to} \frac{\cosh x_-}{\cosh x_+} e^{x\coth 2\tau} \tag{8.4}$$

for function (3.2), cf. Eq. (5.17) in [40]. In correspondence with (3.4), this is a nonphysical eigenstate of H_τ of eigenvalue $-1/\sinh^2 2\tau$. Function $\Delta(x;\tau)$ in the form (4.1) transforms into

$$\Delta(x;\tau) \overset{k\to 1}{\to} \Delta_\tau(x) = \coth 2\tau + \tanh x_- - \tanh x_+, \tag{8.5}$$

while Eq. (4.2) gives, equivalently,

$$\Delta(x;\tau) \overset{k\to 1}{\to} \Delta_\tau(x) = \frac{2}{\sinh 4\tau} + \tanh 2\tau \tanh x_- \tanh x_+. \tag{8.6}$$

[14] The second system, however, is characterized by another phase (3.14) with τ changed for $\tau - \mathbf{K}$.

[15] The states (2.3) for the valence band should be "renormalized" (divided) by a constant $\Theta(\mathbf{K})/\Theta(0)$ to cancel the multiplicative factor that diverges in the limit $\mathbf{K} \to \infty$ in correspondence with (8.1).

The nonperiodic superpotential (gap function) (8.5) corresponds to the Dashen-Hasslacher-Neveu kink-antikink baryons [2]. For the first-order intertwining operator, we have

$$\mathcal{D}(x;\tau)\overset{k\mapsto 1}{\longrightarrow}\frac{d}{dx}-\Delta_\tau(x)\equiv X_\tau, \tag{8.7}$$

cf. (2.26) in [40]. It is the operator that appears in the limit structure of the supercharges S_a,

$$S_1\overset{k\mapsto 1}{\longrightarrow}\begin{pmatrix}0 & X_\tau^\dagger\\ X_\tau & 0\end{pmatrix}\equiv S_{\mathrm{PT},1},\quad S_2\overset{k\mapsto 1}{\longrightarrow}S_{\mathrm{PT},2}=i\sigma_3 S_{\mathrm{PT},1}. \tag{8.8}$$

For the second-order intertwining operator (5.6),

$$\mathcal{A}(x;\tau)\overset{k\mapsto 1}{\longrightarrow}A_{-\tau}A_\tau^\dagger\equiv Y_\tau, \tag{8.9}$$

where $\lim_{\mathbf{K}\to\infty}\mathcal{D}(x+\tau+\frac{1}{2}\mathbf{K};\frac{1}{2}\mathbf{K})=\lim_{\mathbf{K}\to\infty}\mathcal{D}(x+\frac{1}{2}\mathbf{K};-\tau+\frac{1}{2}\mathbf{K})=\frac{d}{dx}-\tanh x_+\equiv A_\tau(x)$, and $A_{-\tau}$ is obtained via the change $\tau\to-\tau$. A limit of the second-order integrals (5.7) is

$$Q_1\overset{k\mapsto 1}{\longrightarrow}\begin{pmatrix}0 & Y_\tau^\dagger\\ Y_\tau & 0\end{pmatrix}\equiv Q_{\mathrm{PT},1},\quad Q_2\overset{k\mapsto 1}{\longrightarrow}Q_{\mathrm{PT},2}=i\sigma_3 Q_{\mathrm{PT},1}, \tag{8.10}$$

cf. Eq. (2.18) in [40]. The first-order operators A_τ and $A_{-\tau}$ also factorize the self-isospectral pair of the PT Hamiltonians, $H_\tau=A_\tau A_\tau^\dagger$, $H_{-\tau}=A_{-\tau}A_{-\tau}^\dagger$, as well as the free particle Hamiltonian, $H_0=A_\tau^\dagger A_\tau=A_{-\tau}^\dagger A_{-\tau}$.

The phases that appear in the action of the intertwining operators $\mathcal{D}(x;\tau)$ and $\mathcal{A}(x;\tau)$ on the superpartner's eigenstates, see Eqs. (3.11) and (5.10), transform into

$$\begin{gathered}e^{i\varphi^{\mathcal{D}}(\alpha,\tau)}\overset{k\mapsto 1}{\longrightarrow}e^{-2i\mathbb{k}\tau}\cdot\frac{-i\mathbb{k}-\coth 2\tau}{\sqrt{\mathbb{k}^2+\coth^2 2\tau}},\\ e^{i\varphi^{\mathcal{A}}(\alpha,\tau)}\overset{k\mapsto 1}{\longrightarrow}e^{-2i\mathbb{k}\tau}.\end{gathered} \tag{8.11}$$

They are associated with the action of the intertwining operators X_τ and Y_τ on the eigenstates of superpartner systems H_τ and $H_{-\tau}$, and appear in the structure of the eigenstates of the first, (8.8), and the second, (8.10), order integrals of the self-isospectral PT system [40].

By employing the relation $2\mathcal{P}(x_+)=\mathcal{D}^\dagger(x;\tau)\mathcal{A}(x;\tau)-\mathcal{A}^\dagger(x;\tau)\mathcal{D}(x;\tau)$ that follows from Eq. (5.14), we find that

$$\mathcal{P}(x_+)\overset{k\mapsto 1}{\longrightarrow}A_\tau\frac{d}{dx}A_\tau^\dagger\equiv Z_\tau, \tag{8.12}$$

cf. (2.24) in [40]. For the limit of the Lax integrals we then get

$$\begin{gathered}L_1\overset{k\mapsto 1}{\longrightarrow}-i\begin{pmatrix}Z_\tau & 0\\ 0 & Z_{-\tau}\end{pmatrix}\equiv L_{\mathrm{PT},1},\\ L_2\overset{k\mapsto 1}{\longrightarrow}L_{\mathrm{PT},2}=\sigma_3 L_{\mathrm{PT},1}.\end{gathered} \tag{8.13}$$

Finally, for a constant $\mathcal{C}(\tau)=\mathrm{ns}2\tau\mathrm{nc}2\tau\mathrm{dn}2\tau$ that appears in the superalgebraic (anti)commutation relations of our system, we obtain

$$\mathcal{C}(\tau)\overset{k\mapsto 1}{\longrightarrow}\coth 2\tau\equiv\mathcal{C}_{2\tau}, \tag{8.14}$$

cf. the first term in Eq. (8.5).

With the described infinite period limit relations, we find a correspondence between the supersymmetric structures in the self-isospectral one-gap Lamé and PT systems. Particularly, applying the infinite period limit to the superalgebraic relations of the self-isospectral Lamé system and making use of the described correspondence, one may immediately reproduce the superalgebraic relations for the self-isospectral PT system (8.3).

The same τ-dependent constant $\mathcal{C}_{2\tau}=\coth 2\tau$ shows up in representation of the superpotential (8.5) and in the superalgebraic structure for the self-isospectral nonperiodic PT system (8.3) due to relation (8.14). Notice, however, that the corresponding functions of the shift parameter, $z(\tau)$ and $\mathcal{C}(\tau)$, which appear in the periodic system, are different. In the next section we will return to this observation.

The infinite period limit of the second-order intertwining operator $\mathcal{Y}(x;\tau)$ may be found by employing relation (5.6),

$$\lim_{K\to\infty}\mathcal{Y}(x;\tau)=-Y_\tau-\mathcal{C}_{2\tau}X_\tau. \tag{8.15}$$

It plays no special role in the supersymmetric structure of the self-isospectral PT system (8.3). Let us, however, shift $x\to x-\tau$ in (8.15) and then take the limit $\tau\to\infty$. Such a double limit procedure applied to the self-isospectral Lamé system $\mathcal{H}$ produces a nonperiodic supersymmetric system $\hat{\mathcal{H}}=\mathrm{diag}(H_{\mathrm{PT}}(x),H_0(x))$ that is composed from the PT system (8.2) and the free particle $H_0=-\frac{d^2}{dx^2}+1$. Operator $\mathcal{Y}(x;\tau)$ in such a limit transforms into the second-order operator $\hat{y}(x)=\frac{d}{dx}(\frac{d}{dx}+\tanh x)$, which intertwines H_{PT} with H_0, $\hat{y}(x)H_{\mathrm{PT}}(x)=H_0(x)\hat{y}(x)$. The kernel of $\hat{y}$ is formed by the singlet eigenstates $1/\cosh x$ $(E=0)$ and $\tanh x$ $(E=1)$ of the PT system $H_{\mathrm{PT}}(x)$, cf. the discussion of the kernel of $\mathcal{Y}(x;\frac{1}{2}\mathbf{K})$ in Sec. V. The Hermitian conjugate operator $\hat{y}^\dagger(x)$ intertwines as $\hat{y}^\dagger(x)H_0(x)=H_{\mathrm{PT}}(x)\hat{y}^\dagger(x)$, and annihilates the eigenstate 1 of the lowest energy $E=1$ and a nonphysical state $\sinh x$ of zero energy in the spectrum of H_0. Integrals S_a, $Q_a^{\mathcal{Y}}$, and L_a transform in such a double limit into the integrals of the supersymmetric system $\hat{\mathcal{H}}$,

$$\begin{gathered}S_1\to-\begin{pmatrix}0 & A_0\\ A_0^\dagger & 0\end{pmatrix}\equiv\hat{s}_1,\qquad Q_1^{\mathcal{Y}}\to\begin{pmatrix}0 & \hat{y}^\dagger\\ \hat{y} & 0\end{pmatrix}\equiv\hat{q}_1^{y},\\ L_1\to-i\begin{pmatrix}A_0\frac{d}{dx}A_0^\dagger & 0\\ 0 & H_0\frac{d}{dx}\end{pmatrix}\equiv\hat{l}_1,\end{gathered} \tag{8.16}$$

and $S_2 \to \hat{s}_2 = i\sigma_3\hat{s}_1$, $Q_2^y \to \hat{q}_2^y = i\sigma_3\hat{q}_1^y$, $L_2 \to \hat{l}_2 = \sigma_3\hat{l}_1$, where $A_0 = \lim_{\tau\to\infty} A_\tau(x-\tau) = \frac{d}{dx} - \tanh x = A_0(x)$, and we have used the relations $\lim_{\tau\to\infty} A_{-\tau}(x) = \frac{d}{dx} + 1$, and $A_0^\dagger A_0 = H_0$, and $\hat{y} = -\frac{d}{dx} A_0^\dagger$.

The nonperiodic superpotential (gap function) $\Delta(x) = \tanh x$ that appears in the structure of the first- and second-order intertwining operators as well as in that of the integrals (8.16) corresponds to the famous Callan-Coleman-Gross-Zee kink solution [2,18,30] of the GN model.

From the total number of seven integrals of motion (6.1) and σ_3, each of which can be used as a grading operator for the self-isospectral Lamé and PT systems, only three integrals survive in the described double limit: in addition to the obvious operator σ_3, nonlocal operators $\mathcal{R}$ and $\mathcal{R}\sigma_3$ are also the integrals for supersymmetric system $\hat{\mathcal{H}}$. The last two operators originate in the double limit from the integrals $\mathcal{R}\mathcal{T}$ and $\mathcal{R}\mathcal{T}\sigma_3$. Having in mind this correspondence, Table I may still be used for the identification of the $\mathbb{Z}_2$ parities of the integrals $\hat{s}_a$, $\hat{q}_a^y$, and $\hat{l}_a$, and it is not difficult to obtain corresponding forms for superalgebra for each of the three possible choices of the grading operator in this case, see [39,48].

Let us look what happens here with the Witten index. As we discussed at the beginning of this section, the only asymmetry between the spectra of the superpartner Hamiltonians $H_{\rm PT}$ and H_0 is the presence of the zero energy bound state in the first superpartner system, which is described by the eigenstate $(1/\cosh x, 0)^T$ of the supersymmetric system $\hat{\mathcal{H}}$. The doublet with $E = 1$ is formed by the eigenstates $(\tanh x, 0)^T$ and $(0, 1)^T$. The first state is an eigenstate of all the three operators σ_3, $\mathcal{R}$, and $\mathcal{R}\sigma_3$ with the same eigenvalue $+1$, while for the second and third states the eigenvalues are, respectively, $+1$, -1, -1, and -1, $+1$, -1. All of the fourth-fold degenerate energy levels in the scattering part of the spectrum with $E > 1$ contribute zero into the Witten index $\Delta_W = \mathrm{Tr}\Gamma$. As a result, for all three choices of the grading operator for the nonperiodic supersymmetric system $\hat{\mathcal{H}}$, we have consistently $|\Delta_W| = 1$.[16]

On the other hand, the first-order matrix operator $\hat{s}_1$ is identified here as a limit of the BdG Hamiltonian $H_{\rm BdG} = S_1$. As may be checked directly, operator $\mathcal{R}\sigma_3$ commutes with $\hat{s}_1$ in accordance with Table I if to take into account the correspondence between nonlocal integrals discussed above. Therefore, it can be identified as a grading operator for a peculiar supersymmetry of the BdG system with the Hamiltonian $\hat{h}_{\rm BdG} = \hat{s}_1$, in which the second-order integral $\hat{q}_2^y$, and the nonlocal operator $i\mathcal{R}\sigma_3\hat{q}_2^y$ are identified as the odd supercharges, and $\hat{l}_1 = -\hat{s}_1\hat{q}_2^y$, cf. (5.36). The corresponding superalgebra has the form (7.5) with obvious substitutions. The state $(1/\cosh x, 0)^T$, is a unique zero mode of the first-order matrix Hamiltonian $\hat{s}_1$, while the two states $(\tanh x, \pm 1)^T$ are the singlet eigenstates of $\hat{s}_1$ of the eigenvalues ± 1, which are also the eigenstates of the grading operator $\mathcal{R}\sigma_3$ of the eigenvalue -1.

Thus, the modulus of the Witten index changes from zero to one for the supersymmetries of both the second $\hat{\mathcal{H}}$ and first $h_{\rm BdG} = \hat{s}_1$ order systems. This reflects effectively the changes in the spectrum that happen in the described infinite period limit of the self-isospectral second-order Lamé and the associated first-order BdG systems.

IX. EXTENDED SUPERSYMMETRIC PICTURE AND DARBOUX DRESSING

Let us now discuss another interesting aspect of our self-isospectral periodic supersymmetric system from the viewpoint of the infinite period limit. As it was shown in [40], the supersymmetric structure of the nonperiodic self-isospectral system (8.3) has a peculiar property: all of its integrals can be treated as a Darboux-dressed form of the integrals of a free particle system $H_0(x)$. We clarify now what corresponds here, in the periodic case, to the Darboux-dressing structure of the self-isospectral PT system (8.3). For that, we extend a picture related to the intertwining operators and the Darboux displacements associated with them.

Consider along with our self-isospectral supersymmetric Lamé system (3.8), $\mathcal{H}(x) = \mathrm{diag}(H(x+\tau), H(x-\tau))$, its copy shifted for the half period, $\mathcal{H}(x+\mathbf{K}) = \mathrm{diag}(H(x+\mathbf{K}+\tau), H(x+\mathbf{K}-\tau))$. Any two of the four (single-component) Hamiltonians may be connected by the intertwining relation of the form $\mathcal{D}(\xi;\mu)H(\xi+\mu) = H(\xi-\mu)\mathcal{D}(\xi;\mu)$. Putting $\xi = x + \frac{1}{2}(\tau_1+\tau_2)$ and $\mu = \frac{1}{2}(\tau_1-\tau_2)$, $\tau_1 \neq \tau_2 + 2\mathbf{K}n$, we present this relation in a more appropriate form,

$$\mathcal{D}\left(x+\tfrac{1}{2}(\tau_1+\tau_2);\tfrac{1}{2}(\tau_1-\tau_2)\right)H(x+\tau_1) = H(x+\tau_2)\mathcal{D}\left(x+\tfrac{1}{2}(\tau_1+\tau_2);\tfrac{1}{2}(\tau_1-\tau_2)\right). \quad (9.1)$$

Here τ_1 and τ_2 take the values in the set $\{-\tau, \tau, -\tau+\mathbf{K}, \tau+\mathbf{K}\}$, and the supersymmetric Hamiltonians $\mathcal{H}(x)$ and $\mathcal{H}(x+\mathbf{K})$ may be related by $\tilde{\mathcal{D}}\mathcal{H}(x+\mathbf{K}) = \mathcal{H}(x)\tilde{\mathcal{D}}$, $\tilde{\mathcal{D}}^\dagger\mathcal{H}(x) = \mathcal{H}(x+\mathbf{K})\tilde{\mathcal{D}}^\dagger$, where

$$\tilde{\mathcal{D}} = \mathrm{diag}\left(\mathcal{D}(x+\tau+\tfrac{1}{2}\mathbf{K};\tfrac{1}{2}\mathbf{K}), \mathcal{D}(x-\tau+\tfrac{1}{2}\mathbf{K};\tfrac{1}{2}\mathbf{K})\right). \quad (9.2)$$

In the general case, if any two Hamiltonians h and $\tilde{h}$ are related by intertwining operators D and $D^\dagger$, $Dh = \tilde{h}D$, $hD^\dagger = D^\dagger\tilde{h}$, and if J is an integral for h, $[h, J] = 0$, then the operator $\tilde{J} = DJD^\dagger$ is an integral for $\tilde{h}$. The system $\mathcal{H}(x)$ is characterized by the set of local integrals of motion $J(x) = \{\sigma_3, S_a(x), Q_a(x), L_a(x)\}$, while the system $\mathcal{H}(x+\mathbf{K})$, is described by the same but shifted set,

[16] Δ_W takes values $+1$ for $\Gamma = \sigma_3$ and $\mathcal{R}$, and -1 for $\mathcal{R}\sigma_3$. A difference in sign is not important, however, since it can be removed by changing a sign in definition of the grading operator in the last case.

$J(x+\mathbf{K})$. Identifying $\mathcal{H}(x+\mathbf{K})$, $\mathcal{H}(x)$, and $\tilde{\mathcal{D}}$ with h, $\tilde{h}$, and D, respectively, we find that $\tilde{J}=\tilde{\mathcal{D}}J(x+\mathbf{K})\tilde{\mathcal{D}}^{\dagger}=J(x)\mathcal{H}(x)$. In other words, the Darboux-dressed integral of one system is just the corresponding integral of another, displaced self-isospectral periodic system, multiplied by its Hamiltonian. Nonlocal operators (6.1), which are the integrals for $\mathcal{H}(x)$, are also the integrals of motion for the displaced system $\mathcal{H}(x+\mathbf{K})$. Then one finds that a similar relation is valid also for these nonlocal integrals as well as for nontrivial diagonal nonlocal integrals (6.11). The only difference is that for all the integrals that contain a factor $\mathcal{R}$, including (6.11), there appears a minus sign, like in $\tilde{\mathcal{D}}\,\tilde{S}(x+\mathbf{K})\tilde{\mathcal{D}}^{\dagger}=-\tilde{S}(x)\mathcal{H}(x)$. Notice also that the Darboux-dressed form of the trivial integral $\mathbb{1}$ (that is a unit two-by-two matrix) for the displaced system $\mathcal{H}(x+\mathbf{K})$ coincides with the Hamiltonian $\mathcal{H}(x)$, $\tilde{\mathcal{D}}\mathbb{1}\tilde{\mathcal{D}}^{\dagger}=\mathcal{H}(x)$.

Since both of the self-isospectral supersymmetric systems are just two copies of the same periodic system shifted mutually in the half period, the described picture is not so unexpected. Let us look, however, at this result from another viewpoint. In the infinite period limit, supersymmetric systems $\mathcal{H}(x)$ and $\mathcal{H}(x+\mathbf{K})$ transform, respectively, into (8.3) and

$$\mathcal{H}_0=\mathrm{diag}(H_0,H_0), \tag{9.3}$$

where $H_0=-\frac{d^2}{dx^2}+1$ is a (shifted for a constant additive term) free particle Hamiltonian. In other words, the infinite period limit of the system $\mathcal{H}(x+\mathbf{K})$ is given by the two copies of the free nonrelativistic particle. As we have seen, the infinite period limit applied to the integrals of the self-isospectral system $\mathcal{H}(x)$ produces corresponding integrals of the self-isospectral PT system (8.3). The infinite period limit of the integrals of the system $\mathcal{H}(x+\mathbf{K})$ may easily be obtained just by taking the limit $x\to\infty$ of the integrals of the self-isospectral PT system (8.3). For nontrivial local integrals, we find

$$\begin{aligned} &S_1(x+\mathbf{K})\to -i\frac{d}{dx}\sigma_2-\mathcal{C}_{2\tau}\sigma_1\equiv s_1,\\ &S_2(x+\mathbf{K})\to s_2=i\sigma_3 s_1, \end{aligned} \tag{9.4}$$

$$\begin{aligned} &Q_a(x+\mathbf{K})\to(-1)^{a+1}\sigma_a\cdot\mathcal{H}_0,\\ &L_1(x+\mathbf{K})\to -i\frac{d}{dx}\cdot\mathcal{H}_0\equiv\ell_1,\\ &L_2(x+\mathbf{K})\to\ell_2=\sigma_3\ell_1. \end{aligned} \tag{9.5}$$

The obtained operators are the integrals of motion for the trivial free particle supersymmetric system (9.3). They correspond to the obvious integrals σ_a, and to the products of them with $-i\frac{d}{dx}$ and $\mathcal{H}_0$. System (9.3) is intertwined with the self-isospectral PT system (8.3) by the infinite period limit of the operator (9.2), $\hat{\mathcal{D}}\to\mathrm{diag}(A_\tau,A_{-\tau})\equiv\mathcal{D}_\infty$, $D_\infty\mathcal{H}_0=\mathcal{H}_{\mathrm{PT}}D_\infty$, $\mathcal{H}_0D_\infty^{\dagger}=D_\infty^{\dagger}\mathcal{H}_{\mathrm{PT}}$. If J_0 is some integral for $\mathcal{H}_0$, then $D_\infty J_0\mathcal{H}_0D_\infty^{\dagger}=D_\infty J_0D_\infty^{\dagger}\mathcal{H}_{\mathrm{PT}}$. Taking into account (9.4) and (9.5), the nontrivial local integrals $S_{\mathrm{PT},a}$, $Q_{\mathrm{PT},a}$, and $L_{\mathrm{PT},a}$ of the self-isospectral PT system (8.3) may be treated as a Darboux-dressed form of the integrals for the free particle system $\mathcal{H}_0$, namely, of s_a, σ_a, and $-iI_a\frac{d}{dx}$, where $I_1=\mathbb{1}$ and $I_2=\sigma_3$.

It is interesting to note that the first-order integral of $\mathcal{H}_0$, for instance, s_1, may also be treated as a Hamiltonian of a free relativistic Dirac particle of mass $\mathcal{C}_{2\tau}$. Then its Darboux-dressed form is a nonperiodic BdG Hamiltonian

$$S_{\mathrm{PT},1}=-i\frac{d}{dx}\sigma_2-\Delta_\tau(x)\sigma_1, \tag{9.6}$$

see Eqs. (8.8) and (8.5). Comparing (9.6) with the structure of s_1 in (9.4), we see that the gap function $\Delta_\tau(x)$ is effectively a Darboux-dressed form of the free Dirac particle's mass $\mathcal{C}_{2\tau}$. The periodic BdG Hamiltonian $H_{\mathrm{BdG}}=S_1$ may be treated then as a periodized form of (9.6), like the Lamé Hamiltonian may be considered as a periodized form of the PT Hamiltonian, see [31]. It is worth stressing, however, that a reconstruction of a crystal structure on the basis of a nonperiodic kink-antikink system is not direct and free of ambiguities: in the previous section we already noted that two different basic functions of the shift parameter in the self-isospectral Lamé and associated BdG systems correspond to the same function in the nonperiodic case.

Another interesting observation can be made on the genesis of the nonlocal integrals (6.11). For the self-isospectral Lamé and PT systems, the reflection operator $\mathcal{R}$ and σ_a, $a=1,2$, are not integrals of motion, but the product of any two of these three operators is an integral of motion. For the supersymmetric free particle system (9.3), however, each of these three operators is an integral of motion. One finds then that the infinite period limit of the integral $\sigma_3\tilde{Q}$, $\sigma_3\tilde{Q}\to\mathrm{diag}(\mathcal{R}Y_\tau,\mathcal{R}Y_{-\tau})\equiv\sigma_3\tilde{Q}_{\mathrm{PT}}$ is exactly a Darboux-dressed form of the reflection operator $\mathcal{R}$, $D_\infty\mathcal{R}D_\infty^{\dagger}=\sigma_3\tilde{Q}_{\mathrm{PT}}$. Or, alternatively, an integral $\tilde{Q}_{\mathrm{PT}}$ for the self-isospectral PT system is a dressed form of the nonlocal diagonal integral $\mathcal{R}\sigma_3$. An analogous relation exists also for the infinite period limit of another nonlocal diagonal integral from (6.11), $D_\infty(-i\mathcal{R}\sigma_2s_1)D_\infty^{\dagger}=\tilde{S}_{\mathrm{PT}}\cdot\mathcal{H}_{\mathrm{PT}}$, where $\tilde{S}_{\mathrm{PT}}=\mathrm{diag}(\mathcal{R}X_\tau,\mathcal{R}X_{-\tau})$.

We conclude that the described Darboux-dressing structure of the self-isospectral PT system, observed earlier in [40], originates from, and is explained by the properties of the self-isospectral periodic one-gap Lamé system.

X. DISCUSSION AND OUTLOOK

To conclude, let us discuss the obtained results from the physics perspective and potential applications and generalizations.

The usual supersymmetric structure of the kink-antikink as well as of the kink crystalline phases of the GN model has been known for about 20 years. However, such a structure with the first-order supercharges and $\mathbb{Z}_2$ grading provided by the diagonal Pauli matrix does not explain or reflect the peculiar, finite-gap nature of the corresponding solutions. It also does not reflect the restoration of the discrete chiral symmetry at the zero value of the bare mass in the GN model, when the kink-antikink crystalline condensate transforms into the kink crystal. Both aspects are explained by the exotic nonlinear supersymmetric structure we revealed here. The finite-gap nature is reflected by the Lax integral incorporated into a nonlinear supersymmetric structure alongside the first- and second-order supercharges. A restoration of the discrete chiral symmetry, on the other hand, is reflected by structural changes that happen in nonlinear supersymmetry at the half period shift of the Lamé superpartner systems, when a central gap in the spectrum of the associated BdG system disappears. We showed that the first-order BdG system [17] has its own supersymmetry, which can be revealed only with the help of the nonlocal grading operators investigated in Sec. VI. The disappearance of the middle gap in the BdG spectrum is accompanied by emergence of the new, nontrivial second-order integral of motion in the first-order system (while the BdG Hamiltonian has no such integral in the kink-antikink crystalline phase).

The aspects related to the infinite period limit we investigated in Secs. VIII and IX may be useful for understanding of some puzzles related to a computation of the Witten index in some supersymmetric field theories when a system is put in a periodized box [49].

Recently, perfect Klein tunneling in carbon nanostructures was explained in [50] by an unusual supersymmetric structure with the first-order matrix Hamiltonian. We believe that the supersymmetry we investigated here, particularly in Sec. VII, may also be useful in the study of other phenomena in graphene, where the dynamics of charges is governed by the effective first-order Dirac Hamiltonian.

It would be interesting to clarify whether the twisted kink crystal of the GN model with continuous chiral symmetry, that was found in [7,8], could be obtained by supersymmetric constructions similar to those in Sec. III.

We treated λ, which appears in the structure of the second-order intertwining operator $\mathcal{B}(x;\tau,\lambda)$ of a general form (5.3), as a kind of a virtual shift parameter. One could extend the picture by reinterpreting Eqs. (5.1) and (5.2) as intertwining relations for the three Lamé systems, $H(x+\tau_1)$, $H(x+\tau_2)$, and $H(x+\tau_3)$, where $\tau_1=\tau$, $\tau_2=\tau+2\lambda$, and $\tau_3=-\tau$. Then we would get an extended self-isospectral system of three superpartner Lamé Hamiltonians. Employing a relation of the form (9.1), one could further extend the picture to obtain a self-isospectral system with $n>3$ superpartners $H(x+\tau_1),\ldots,H(x+\tau_n)$. When the shift parameters are such that $\tau_n=\tau_1$, the corresponding intertwining operator of order n would reduce to an integral for the system $H(x+\tau_1)$. It is in such a way that we identified, in fact, the third-order Lax operator $\mathcal{P}(x+\tau)$ for the system $H(x+\tau)$. The interesting questions that arise are, what is a complete set of integrals and what kind of supersymmetry do we get for such an n-component self-isospectral system? Particularly, what is the nature of the above-mentioned integral of motion of the order n for $n>3$? What is the relation of such extended supersymmetric systems with the GN model and what physics could be associated with them?

[17] It is this first-order system that really describes the corresponding crystalline phases in the GN model, while the second-order Lamé system is related to it as the Klein-Gordon equation is related to the Dirac equation.

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APPENDIX A: JACOBI ELLIPTIC FUNCTIONS

We summarize here some properties and relations for Jacobi elliptic and related functions. For details, see, e.g., [29,51].

In notations for these functions we suppress a dependence on a modular parameter $0<k<1$, $\mathrm{sn}x=\mathrm{sn}(x|k)$, etc., when this does not lead to ambiguities. On the other hand, a dependence on a complementary modulus parameter $0<k'<1$, $k'=(1-k^2)^{1/2}$, is indicated explicitly. We use Glaisher's notation for inverse quantities and quotients of Jacobi elliptic functions, $\mathrm{nd}x=1/\mathrm{dn}x$, $\mathrm{ns}x=1/\mathrm{sn}x$, $\mathrm{nc}x=1/\mathrm{cn}x$, $\mathrm{sc}x=\mathrm{sn}x/\mathrm{cn}x$, etc.

The basic Jacobi elliptic functions are the doubly periodic meromorphic functions $\mathrm{sn}u$, $\mathrm{cn}u$, and $\mathrm{dn}u$, whose periods are $(4\mathbf{K},2i\mathbf{K}')$, $(4\mathbf{K},2\mathbf{K}+2i\mathbf{K}')$ and $(2\mathbf{K},4i\mathbf{K}')$, respectively. $\mathrm{sn}u$ is an odd function, while $\mathrm{cn}u$ and $\mathrm{dn}u$ are even functions, which are related by the identities $\mathrm{sn}^2u+\mathrm{cn}^2u=1$, $\mathrm{dn}^2u+k^2\mathrm{sn}^2u=1$, $k^2\mathrm{cn}^2u+k'^2=\mathrm{dn}^2u$, $k'^2\mathrm{sn}^2u+\mathrm{cn}^2u=\mathrm{dn}^2u$, and whose derivatives are $\frac{d}{du}\mathrm{sn}u=\mathrm{cn}u\mathrm{dn}u$, $\frac{d}{du}\mathrm{cn}u=-\mathrm{sn}u\mathrm{dn}u$, $\frac{d}{du}\mathrm{dn}u=-k^2\mathrm{sn}u\mathrm{cn}u$. They have simple zeros and poles at

$$\mathrm{sn}u:\ 0,2\mathbf{K};\quad \mathrm{cn}u:\ \mathbf{K},-\mathbf{K};\quad \mathrm{dn}u:\ \mathbf{K}+i\mathbf{K}',\mathbf{K}-i\mathbf{K}', \tag{A1}$$

$$\mathrm{sn}\,u,\mathrm{cn}u:\ i\mathbf{K}',2\mathbf{K}+2i\mathbf{K}';\qquad \mathrm{dn}u:\ i\mathbf{K}',-i\mathbf{K}', \tag{A2}$$

respectively, modulo periods. Here

$$\mathbf{K} = \mathbf{K}(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{A3}$$

is a complete elliptic integral of the first kind, and $\mathbf{K}' = \mathbf{K}(k')$ is a complementary integral, which are monotonic functions of k in the interval $0 < k < 1$: $d\mathbf{K}/dk > 0$, $d\mathbf{K}'/dk < 0$. In the limit cases $k = 0$ and $k = 1$, elliptic functions transform into simply-periodic functions in a complex plane,

$$\begin{gathered} k = 0, k' = 1: \mathbf{K} = \tfrac{1}{2}\pi, \mathbf{K}' = \infty, \\ \mathrm{sn}u = \sin u, \quad \mathrm{cn}u = \cos u, \qquad \mathrm{dn}u = 1, \end{gathered} \tag{A4}$$

$$\begin{gathered} k = 1, k' = 0: \mathbf{K} = \infty, \mathbf{K}' = \frac{1}{2}\pi, \\ \mathrm{sn}u = \tanh u, \quad \mathrm{cn}u = \mathrm{dn}u = \frac{1}{\cosh u}. \end{gathered} \tag{A5}$$

The addition formulae are

$$\begin{aligned} \mathrm{s}_+ &= \frac{1}{\mu}(\mathrm{s}_u\mathrm{c}_v\mathrm{d}_v + \mathrm{s}_v\mathrm{c}_u\mathrm{d}_u), \\ \mathrm{c}_+ &= \frac{1}{\mu}(\mathrm{c}_u\mathrm{c}_v - \mathrm{s}_u\mathrm{s}_v\mathrm{d}_u\mathrm{d}_u), \\ \mathrm{d}_+ &= \frac{1}{\mu}(\mathrm{d}_u\mathrm{d}_v - k^2\mathrm{s}_u\mathrm{s}_v\mathrm{c}_u\mathrm{c}_u), \end{aligned} \tag{A6}$$

where $\mathrm{s}_+ = \mathrm{sn}(u+v)$, $\mathrm{s}_u = \mathrm{sn}u$, $\mathrm{s}_v = \mathrm{sn}v$, $\mathrm{c}_+ = \mathrm{cn}(u+v)$, $\mathrm{d}_+ = \mathrm{dn}(u+v)$, etc., and $\mu = 1 - k^2\mathrm{sn}^2u\mathrm{sn}^2v$. Jacobi's imaginary transformation is

$$\begin{gathered} \mathrm{sn}(iu|k) = i\mathrm{sn}(u|k')\mathrm{nc}(u|k'), \quad \mathrm{cn}(iu|k) = \mathrm{nc}(u|k'), \\ \mathrm{dn}(iu|k) = \mathrm{dn}(u|k')\mathrm{nc}(u|k'). \end{gathered} \tag{A7}$$

From the addition formulae and (A7), one finds some displacement properties of Jacobi elliptic functions, which are shown in Table II.

APPENDIX B: JACOBI ZETA, THETA, AND ETA FUNCTIONS

The complete elliptic integral of the second kind is defined by

$$\mathbf{E} = \mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}dx. \tag{B1}$$

It is a monotonically decreasing function, $d\mathbf{E}/dk < 0$. The complete elliptic integrals $\mathbf{K} = \mathbf{K}(k)$ and $\mathbf{E} = \mathbf{E}(k)$ satisfy the first-order differential equations $\frac{d\mathbf{K}}{dk} = \frac{\mathbf{E}-k'^2\mathbf{K}}{kk'^2}$, $\frac{d\mathbf{E}}{dk} = \frac{\mathbf{E}-\mathbf{K}}{k}$, from which an inequality $k'^2 < \mathbf{E}/\mathbf{K} < 1$ and the Legendre's relation $\mathbf{E}\mathbf{K}' + \mathbf{E}'\mathbf{K} - \mathbf{K}\mathbf{K}' = \frac{1}{2}\pi$ may be deduced, where $\mathbf{E}' = \mathbf{E}(k')$ is a complementary integral of the second kind.

The incomplete elliptic integral of the second kind is defined as

$$\mathrm{E}(u) = \int_0^u \mathrm{dn}^2u du, \tag{B2}$$

in terms of which $\mathbf{E} = \mathrm{E}(\mathbf{K})$. This is an odd analytic function of u, regular save for simple poles of residue $+1$ at the points $2n\mathbf{K} + (2m+1)i\mathbf{K}'$. Function $\mathrm{E}(u)$ is not an elliptic function. It possesses the properties of pseudoperiodicity, $\mathrm{E}(u+2\mathbf{K}) - \mathrm{E}(u) = \mathrm{E}(2\mathbf{K}) = 2\mathbf{E}$, $\mathrm{E}(u+2i\mathbf{K}') - \mathrm{E}(u) = \mathrm{E}(2i\mathbf{K}')$, where in the first relation the second equality is obtained by putting $u = -\mathbf{K}$.

In terms of $\mathrm{E}(u)$, a simply periodic Jacobi Zeta function is defined,

$$\mathrm{Z}(u) = \mathrm{E}(u) - \frac{\mathbf{E}}{\mathbf{K}}u, \tag{B3}$$

which satisfies relations $\frac{d\mathrm{Z}(u)}{du} = \mathrm{dn}^2u - \frac{\mathbf{E}}{\mathbf{K}}$, and

$$\begin{gathered} \mathrm{Z}(u+2\mathbf{K}) = \mathrm{Z}(u), \quad \mathrm{Z}(u+2i\mathbf{K}') = \mathrm{Z}(u) - i\frac{\pi}{\mathbf{K}}, \\ \mathrm{Z}(-u) = -\mathrm{Z}(u), \quad \mathrm{Z}(\mathbf{K}-u) = -\mathrm{Z}(\mathbf{K}+u), \end{gathered} \tag{B4}$$

$$\mathrm{Z}(0) = \mathrm{Z}(\mathbf{K}) = 0, \qquad \mathrm{Z}(\mathbf{K}+i\mathbf{K}') = -i\frac{\pi}{2\mathbf{K}}. \tag{B5}$$

Zeta function satisfies an addition formula

$$\mathrm{Z}(u+v) = \mathrm{Z}(u) + \mathrm{Z}(v) - k^2\mathrm{sn}u\mathrm{sn}v\mathrm{sn}(u+v), \tag{B6}$$

and obeys Jacobi's imaginary transformation

$$i\mathrm{Z}(iu|k) = \mathrm{Z}(u|k') + \frac{\pi u}{2\mathbf{K}\mathbf{K}'} - \mathrm{dn}(u|k')\mathrm{sc}(u|k'), \tag{B7}$$

from which one finds $\mathrm{Z}(u+i\mathbf{K}') = \mathrm{Z}(u) + \mathrm{ns}u\mathrm{cn}u\mathrm{dn}u - i\frac{\pi}{2\mathbf{K}}$. For the limit values of the modular parameter, $k = 0$ and $k = 1$, we have

$$\mathrm{Z}(u|0) = 0, \qquad \mathrm{Z}(u|1) = \tanh u. \tag{B8}$$

In terms of $\mathrm{Z}(u) = \mathrm{Z}(u|k)$, the Jacobi Theta function $\Theta(u|k)$ is defined as

$$\Theta(u) = \Theta(0)\exp\left(\int_0^u \mathrm{Z}(u)du\right). \tag{B9}$$

TABLE II. Displacement properties of Jacobi elliptic functions.

u	$u+\mathbf{K}$	$u+i\mathbf{K}'$	$u+\mathbf{K}+i\mathbf{K}'$	$u+2\mathbf{K}$	$u+2i\mathbf{K}'$	$u+2(\mathbf{K}+i\mathbf{K}')$
$\mathrm{sn}u$	$\mathrm{cn}u\mathrm{dn}u$	$\frac{1}{k}\mathrm{ns}u$	$\frac{1}{k}\mathrm{dn}u\mathrm{nc}u$	$-\mathrm{sn}u$	$\mathrm{sn}u$	$-\mathrm{sn}u$
$\mathrm{cn}u$	$-k'\mathrm{sn}u\mathrm{nd}u$	$-i\frac{1}{k}\mathrm{dn}u\mathrm{ns}u$	$-i\frac{k'}{k}\mathrm{nc}u$	$-\mathrm{cn}u$	$-\mathrm{cn}u$	$\mathrm{cn}u$
$\mathrm{dn}u$	$k'\mathrm{nd}u$	$-i\mathrm{cn}u\mathrm{ns}u$	$ik'\mathrm{sn}u\mathrm{nc}u$	$\mathrm{dn}u$	$-\mathrm{dn}u$	$-\mathrm{dn}u$

TABLE III. Parity and some displacement properties of Jacobi Θ and H functions.

u	$-u$	$u+2\mathbf{K}$	$u+i\mathbf{K}'$	$u+2i\mathbf{K}'$	$u+\mathbf{K}+i\mathbf{K}'$	$u+2\mathbf{K}+2i\mathbf{K}'$
$\Theta(u)$	$\Theta(u)$	$\Theta(u)$	$iM(u)\mathrm{H}(u)$	$-N(u)\Theta(u)$	$M(u)\mathrm{H}(u+\mathbf{K})$	$-N(u)\Theta(u)$
$\mathrm{H}(u)$	$-\mathrm{H}(u)$	$-\mathrm{H}(u)$	$iM(u)\Theta(u)$	$-N(u)\mathrm{H}(u)$	$M(u)\Theta(u+\mathbf{K})$	$N(u)\mathrm{H}(u)$

This is an even, $\Theta(-u)=\Theta(u)$, integral periodic function of period $2\mathbf{K}$, whose only zeros are simple ones at the points of the set $2n\mathbf{K}+(2m+1)i\mathbf{K}'$. It satisfies the relation $\Theta(u+2i\mathbf{K}')=-\frac{1}{q}\exp(-i\frac{\pi}{\mathbf{K}}u)\Theta(u)$, where $q=\exp(-\pi\mathbf{K}'/\mathbf{K})$. Notice that sometimes Jacobi's Theta function is defined by the Fourier series,

$$\Theta(u|k)=\vartheta_4(v),\qquad \vartheta_4(z)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz),\qquad v=\frac{\pi u}{2\mathbf{K}}. \tag{B10}$$

Then the Z function can be defined by the logarithmic derivative,

$$\mathrm{Z}(u)=\frac{d}{du}\ln\Theta(u). \tag{B11}$$

In correspondence with definition (B10), a constant in (B9) is fixed as $\Theta(0)=\sqrt{\frac{2\mathbf{K}k'}{\pi}}$.

The Jacobi Eta function $\mathrm{H}(u)$ is defined in terms of the Theta function,

$$\mathrm{H}(u)=-iq^{1/4}\exp\left(i\frac{\pi u}{2\mathbf{K}}\right)\Theta(u+i\mathbf{K}'). \tag{B12}$$

This is an odd, $\mathrm{H}(-u)=-\mathrm{H}(u)$, integral periodic function of period $4\mathbf{K}$, which possesses simple zeros at the points of the set $2n\mathbf{K}+2mi\mathbf{K}'$. Some of the properties of the Eta and Theta functions are summarized in Table III, where $M(u)=\exp(-i\frac{\pi u}{2\mathbf{K}})q^{-1/4}$, $N(u)=\exp(-i\frac{\pi u}{\mathbf{K}})q^{-1}$. For particular values of the argument, we also have $\mathrm{H}'(0)=\frac{\pi}{2\mathbf{K}}\mathrm{H}(\mathbf{K})\Theta(0)\Theta(\mathbf{K})$, $\Theta(\mathbf{K})=\sqrt{\frac{2\mathbf{K}}{\pi}}$, $\mathrm{H}(\mathbf{K})=\sqrt{\frac{2k\mathbf{K}}{\pi}}$. The Jacobi Theta function satisfies a kind of addition theorem,

$$\Theta(u+v)\Theta(u-v)\Theta^2(0)=\Theta^2(u)\Theta^2(v)-\mathrm{H}^2(u)\mathrm{H}^2(v). \tag{B13}$$

The basic Jacobi elliptic functions may be represented in terms of Θ and H functions,

$$\begin{aligned}\mathrm{sn}u&=\frac{\mathrm{H}(u)}{\Theta(u)}\cdot\frac{\Theta(0)}{\mathrm{H}'(0)},\\ \mathrm{cn}u&=\frac{\mathrm{H}(u+\mathbf{K})}{\Theta(u)}\cdot\frac{\Theta(0)}{\mathrm{H}(\mathbf{K})},\\ \mathrm{dn}u&=\frac{\Theta(u+\mathbf{K})}{\Theta(u)}\cdot\frac{\Theta(0)}{\Theta(\mathbf{K})}.\end{aligned} \tag{B14}$$

Under complex conjugation, all the Jacobi elliptic functions as well as H, Θ, and Z satisfy the relation $(f(z))^*=f(z^*)$.

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Capítulo 3

Defectos solitónicos en la jerarquía de ecuacioes de KdV, propagación de solitones en fondo libre y fondo 1-gap

3.1. Evolución en soluciones de la ecuación de KdV

3.1.1. El problema espectral auxiliar asociado al par de Lax de KdV

Considere el sistema lineal

$$L\phi = \lambda\phi\,, \qquad \frac{\partial\phi}{\partial t} = P\phi\,, \tag{3.1}$$

para una función $\phi = \phi(x,t,\lambda)$. Es asumido que L y P son algunos (en general, matriciales) operadores diferenciales en el espacio de cordenadas $x \in \mathbb{R}$ con coeficientes que pueden depender solo en la variable t. Si la evolucion en t generada por P es iso-espectral, $d\lambda/dt = 0$, entonces la condición de consistencia para el sistema (3.1) se reduce a la ecuación de Lax

$$\frac{\partial L}{\partial t} = [P, L]\,. \tag{3.2}$$

Para la elección del par de Lax en la forma de los operadores diferenciales

$$L = -\partial_x^2 + u\,, \tag{3.3}$$

$$P = -4\partial_x^3 + 6u\partial_x + 3u_x\,, \tag{3.4}$$

la ecuación (3.2) se reduce a la ecuacón de Korteweg-de Vries para el campo escalar $u = u(x,t)$,

$$u_t = 6uu_x - u_{xxx}\,. \tag{3.5}$$

3.1.2. Covarianza de la ecuación de KdV bajo transformaciones de Darboux

El sistema de ecuaciones correspondiente al par de Lax (3.3), (3.4),

$$(-\partial_x^2 + u)\Psi(x,t,\lambda) = \lambda\Psi(x,t,\lambda)\,, \tag{3.6}$$

$$\frac{\partial}{\partial t}\Psi(x,t,\lambda) = (-4\partial_x^3 + 6u\partial_x + 3u_x)\Psi(x,t,\lambda)\,, \tag{3.7}$$

es covariante bajo transformaciones de Darboux [53]

$$u(x,t) \quad \rightarrow \quad u_1(x,t) = u(x,t) - 2(\log \Psi(x,t,\lambda_1))_{xx}\,, \tag{3.8}$$

$$\Psi(x,t,\lambda) \quad \rightarrow \quad \Psi_1(x,t,\lambda) = \frac{W(\Psi(x,t,\lambda_1),\Psi(x,t,\lambda))}{\Psi(x,t,\lambda_1)}\,, \tag{3.9}$$

acá W es el Wronskiano, $W(f,g) = fg_x - f_x g$. Esto sigue de la observación que si (3.6) y (3.7) son cumplidas por $\Psi(x,t,\lambda)$, y $\Psi(x,t,\lambda_1)$ satisface la misma ecuación con λ cambiado por λ_1, entonces $\Psi_1(x,t,\lambda)$ obedece la ecuación (3.6), (3.7) con $u(x,t)$ cambiado por $u_1(x,t)$. Como una consecuencia, si $u(x,t)$ es una solución de la ecuación de KdV (5.13), entonces $u_1(x,t)$ obedece la misma ecuaión.

Este resultado puede ser extendido para una secuencia finita de consecutivas transformaciones de Darboux,

$$u(x,t) \quad \rightarrow \quad u_m(x,t) = u(x,t) - 2(\log W(\Psi(x,t,\lambda_1),\ldots,\Psi(x,t,\lambda_m)))_{xx}\,, \tag{3.10}$$

$$\Psi(x,t,\lambda) \quad \rightarrow \quad \Psi_m(x,t,\lambda) = \frac{W(\Psi(x,t,\lambda_1),\ldots,\Psi(x,t,\lambda_m),\Psi(x,t,\lambda))}{W(\Psi(x,t,\lambda_1),\ldots,\Psi(x,t,\lambda_m))}\,, \tag{3.11}$$

3.1.3. Soluciones multisolitónicas de la ecuacón de KdV

Describiremos el método para el ejemplo más simple, partiendo de la solución trivial estacionaria de KdV $u_0 = 0$. En este caso $L_0 = -\frac{\partial^2}{\partial x^2}$ corresponde al operador de Schrödinger de la particula libre, y el operador de evolución (3.4) se reduce a $P_0 = -4\frac{\partial^3}{\partial x^3}$. El sistema (3.1) toma entonces la forma

$$-\frac{\partial^2\Psi}{\partial x^2} = \lambda\Psi\,, \qquad \frac{\partial\Psi}{\partial t} = -4\frac{\partial^3\Psi}{\partial x^3}\,. \tag{3.12}$$

Actuando en ambos lados por $4\partial_x$ y utilizando la segunda, ecuación obtenemos $\partial_t\Psi = 4\lambda\partial_x\Psi$. Por lo que, $\Psi(x,t,\lambda) = \Psi(x+4\lambda t,\lambda)$. Para $\lambda = 0$, $\lambda = -\kappa^2 < 0$, y $\lambda = \kappa^2 > 0$, el par de soluciones linealmente independientes del sistema (3.12) pueden ser elegidas en la forma

$$\Psi(x,t,\lambda=0) = \{1,\, x\}\,, \tag{3.13}$$

$$\Psi(x,t,\lambda=-\kappa^2) = \{\cosh X^-,\, \sinh X^-\}, \qquad \Psi(x,t,\lambda=\kappa^2) = \{\cos X^+,\, \sin X^+\}\,, \tag{3.14}$$

acá $X^\mp = \kappa(x - x_0 \mp 4\kappa^2 t)$. Al aplicar la transformación de Crum-Darboux a la solución trivial $u_0 = 0$, tenemos diferentes posibilidades de elegir funciones de onda dentro del conjunto (3.13), (3.14). la elección $\Psi(x,t,0) = x$ nos da la solución estacionaria no trivial más simple pero singular de la ecuación de KdV, $u_1(x) = 2/x^2$. Las soluciones *no singulares* de la ecuación de KdV son generadas eligiendo apropiadamente los estados con $\lambda < 0$,

$$u_n(x,t) = -2\frac{\partial^2}{\partial x^2}\log W(\cosh X_1^-, \sinh X_2^- \ldots, f(X_n^-))\,, \qquad X_j^- = \kappa_j(x - x_{0j} - 4\kappa_j^2 t)\,, \tag{3.15}$$

acá el último argumento en el Wronskiano es $f(X_n^-) = \sinh X_n^-$ si n es par, $n = 2l$, y $f(X_n^-) = \cosh X_n^-$ para n impar $n = 2l+1$; x_{0j} son parámetros de desfase, y los parámetros de escala κ_j

deben obedeser las inequaciones $0 < \kappa_1 < \kappa_2 < \kappa_3 < \ldots < \kappa_n$. Las funciones (3.15) corresponden a soluciones n-solitónicas dela ecuación de KdV. El caso $n = 2$ es ilustrado en la Fig. 3.1. Cuando los solitones de la solución (3.15) estan lo suficientemente separados, la propagación ha-

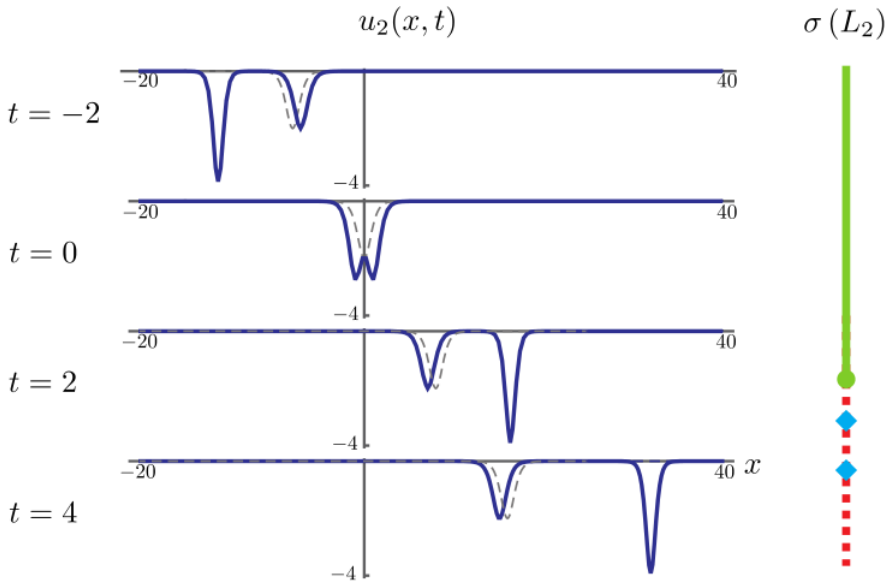

Figura 3.1: La solución dos-solitónica de KdV con $\kappa_1 = 1$, $\kappa_2 = 1{,}4$, y $x_{0i} = 0$, $i = 1, 2$, es mostrada en linea continua. La propagación de la solución solitón $\kappa_1 = 1$ y $x_{01} = 0$ es descrita en linea entrecortada. Las fases iniciales han sido elegidas de tal manera que las soluciones sean simétricas con respecto al punto $x = 0$ para $t = 0$. A la derecha se muestra el espectro del operador de Schrödinger L_2 con potencial dos-solitónico. La linea verde continua corresponde a la parte continua semi infinita doblemente degenerada del espectro con auto estados $\psi^{\pm\kappa}(x,t) = \mathbb{A}_2 e^{\pm iX^+(x,t;\kappa,x_0)}$, mientras que el circulo en el borde corresponde al estado no degenerado en el borde de la banda descrito por el auto estado $\psi_0(x,t) = \mathbb{A}_2 1$. La linea roja entrecortada corresponde a la parte no física del espectro, to non-physical semi-infinite part of the spectrum, los cuadrados azules corresponden a los estados ligados atrapados por los solitones y corresponden a los estados $\psi_1(x,t) = \mathbb{A}_2 \sinh X_1^-$ y $\psi_2(x,t) = \mathbb{A}_2 \cosh X_2^-$.

cia la derecha del j-esimo solitón puede ser caracterizado por la velocidad $\mathcal{V}_j = 4\kappa_j^2$ y la amplitud $2\kappa_j^2$.

3.2. Solitones en fondo cnoidal

La solución estacionaria y periódica de la ecuación de KdV (5.13) puede ser representada en la forma

$$u(x) = u_{0,0}(x) = 2k^2\mu^2 \mathsf{sn}^2\left(\mu x\,|k\right) - \frac{2}{3}\left(1+k^2\right)\mu^2\,, \tag{3.16}$$

acá $\mathsf{sn}\,(u|k)$ es la función elíptica de Jacobi, cuyos periodos real e imaginario dependen en el parámetro modular $0 < k < 1$, $\mu > 0$ es un parámetro libre (escala). Debido a la independencia en el tiempo t de esta solución (3.16), la ecuación de Lax (3.2) se reduce a la condición de conmuitatividad de los operadores (3.3) y (3.4) construidos en base a (3.16),

$$[L, P] = 0\,. \tag{3.17}$$

La relación (3.17) garantiza la existencia de una base común de autoestados para los operadores L y P. Buscaremos soluciones para el sistema de ecuaciones (3.6), (3.7) en la forma

$$\Psi(x,t,\lambda) = \Phi(x,\alpha)\exp(\pi(\alpha)t)\,, \tag{3.18}$$

acá $\Phi(x,\alpha)$ es un auto estado común de L y P, $L\Phi(x,\alpha) = \lambda(\alpha)\Phi(x,\alpha)$, $P\Phi(x,\alpha) = \pi(\alpha)\Phi(x,\alpha)$. El estado buscado $\Phi(x,\alpha)$ es [19, 41, 4]

$$\Phi(x,\alpha) = \frac{\mathrm{H}\left(\mu x + \alpha|k\right)}{\Theta\left(\mu x|k\right)} e^{-\mu x \mathrm{Z}(\alpha|k)}, \tag{3.19}$$

acá H, Θ y Z son las funciones Eta, Theta y Zeta de Jacobi, mientras los correspondientes auto-estdos son

$$\lambda(\alpha|k) = \mu^2\left(\mathrm{dn}^2\big(\alpha|k\big) - \frac{1}{3}(1+k'^2)\right), \tag{3.20}$$

$$\pi(\alpha|k) = -4k^2\mu^3 \mathsf{sn}\left(\alpha\,|k\right)\mathsf{cn}\left(\alpha\,|k\right)\mathsf{dn}\left(\alpha\,|k\right)\,. \tag{3.21}$$

Note que $\lambda(-\alpha) = \lambda(\alpha)$, $\pi(-\alpha) = -\pi(\alpha)$, $\pi(\alpha) = 2\mu\frac{d\lambda(\alpha)}{d\alpha}$, y

$$\pi^2(\alpha) = -16\big(\lambda(\alpha) - E_0\big)(\lambda(\alpha) - E_1)(\lambda(\alpha) - E_2)\,, \tag{3.22}$$

acá

$$E_0 = -\frac{1}{3}(1+k'^2)\mu^2\,, \qquad E_1 = \frac{1}{3}(1-2k^2)\mu^2\,, \qquad E_2 = \frac{1}{3}(1+k^2)\mu^2\,, \tag{3.23}$$

y $k' = \sqrt{1-k^2}$ es el parámetro modular complementario.

El espectro del operador de Schrödinger 3.6 con potencial cnoidal 3.16 está dividido en dos bandas prohibidas y dos bandas permitidas siendo E_0, E_1 y E_2 las energías de los vordes de banda, La existencia de estas dos bandas prohibidas permite definir dos tipos distintos de estados no físicos (uno con infinitos ceros y otros con uno o ningun cero) que se ven reflejados en dos tipos distintos de defectos solitónicos para el potencial de Lamé. Ambos tipos de defectos se diferencian por la dirección en que se propagan y por su forma, defectos construidos a partir de estados

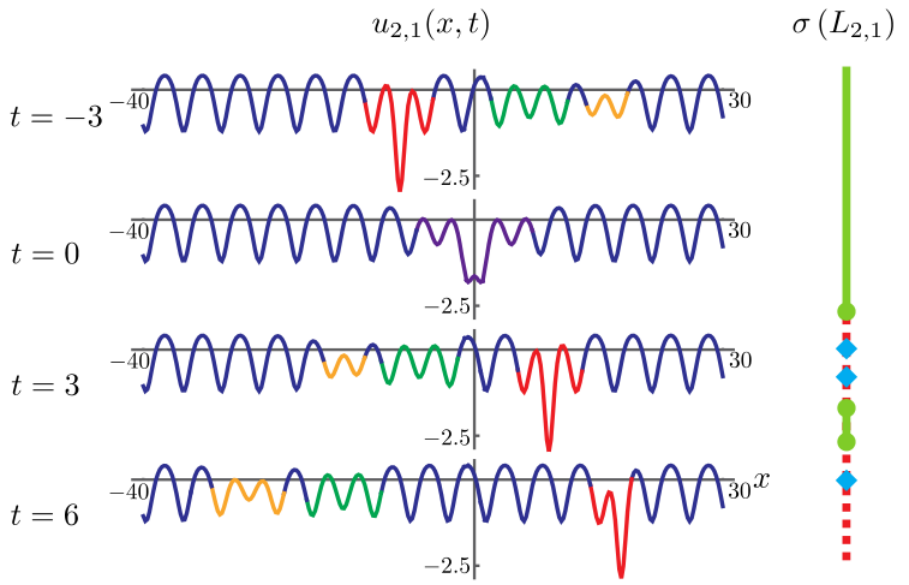

Figura 3.2: Se muestra una solución de la ecuación de KdV con un solitón de tipo pulso que se mueve a la derecha y dos solitones tipo compresión que se mueven a la izquierda. A la derecha se muestra el espectro de $L_{2,1} = -\frac{d^2}{dx^2} + u_{2,1}(x,t)$, el solitón tipo pulso soporta el estado ligado de más baja energía, los otros dos estados ligados en la banda interior estan soportados por los solitones tipo compresión o modulación y la banda finita permitida es debido a la estructura cristalina del potencial inicial. Para el tiempo $t = 0$, los solitones estan en la zona de mayor interacción. Los parámetros han sido elegidos de tal forma que la solución para el tiempo $t = 0$ es simétrica con respecto al punto $x = 0$.

no físicos de la banda prohibida semi-infinita inferior tienen forma de pulso y se propagan a la derecha, mientras que los defectos construidos a partir de estados no físicos de la banda interior modulan la red cristalina y se propagan a la izquierda. El estudio completo de la construcción y evolución de estos defectos fué estudiada en [9] en la Fig. 3.2 se muestra una solución mixta con ambos tipos de defectos para la ecuación de KdV.

3.3. Defectos solitónicos en la jerarquía estacionaria de KdV

En la siguiente publicación los potenciales solitónicos y sus respectivas integrales de Lax-Novikov han sido estudiados en detalle para fondo libre y fondo periódico o de Lamé. Estableciendo normas concretas a partir del teorema de ceros de la construcción de potenciales solitónicos en fondo del potencial de Lamé [5, 4]. También acá fueron estudiadas las superpercargas de orden uno de la supersimetría extendida $N = 4$ con coeficientes no lineales dada su relevante

interpretación como operador de Dirac u operador de Bogoliubov-de Gennes, cuadro en que el superpotencial cumple el rol de potencial transparente escalar para el operador de Dirac en una dimensión o condensado no lineal de fermiones para el modelo de Gross-Neveu según el sistema físico correspondiente. Para efecto del siguiente texto debemo entender como se agrega un solitón a un potencial en la forma de Its-Matveev para eso introducimos las transformaciones de Darboux solitónicas

Transformaciones de Darboux solitónicas: construidas a partir de estados $\psi_{a_1,a_2}(r_1,r_2,x) = a_1\psi(r_1,x) + a_2\psi(r_2,x)$, acá a, b son $\mathbb{C}$ constantes y r_1 y r_2 son elementos en cartas distintas de la superficie de Riemann relacionada a la curva compleja hiper-elíptica 1.57 con $z(r_1) = z(r_2)$ y $y(r_1) \neq y(r_2)$, el cual produce los cambios

$$\begin{aligned} u(x) &= -2\tfrac{d^2}{dx^2}\ln(\theta(x\mathbf{v}+\boldsymbol{\phi},\tau)) + \Lambda_0 \\ \to \hat{u}(x) &= -2\tfrac{d^2}{dx^2}\ln(a_1\theta(x\mathbf{v}+\boldsymbol{\phi}+\boldsymbol{\alpha}(r_1),\tau)\exp(-ix\xi(r_1)) \\ &\quad + a_2\theta(x\mathbf{v}+\boldsymbol{\phi}+\boldsymbol{\alpha}(r_2),\tau)\exp(-ix\xi(r_2))) + \Lambda_0, \end{aligned} \tag{3.24}$$

El operador

$$A = \psi_{a_1,a_2}(r_1,r_2,x)\frac{d}{dx}\frac{1}{\psi_{a_1,a_2}(r_1,r_2,x)}, \quad A^{\#} \equiv -\frac{1}{\psi_{a_1,a_2}(r_1,r_2,x)}\frac{d}{dx}\psi_{a_1,a_2}(r_1,r_2,x), \tag{3.25}$$

entrelaza $H = A^{\#}A + z(r_1)$ y $\hat{H} = AA^{\#} + z(r_1) = -\frac{d^2}{dx^2} + \hat{u}(x)$, en la forma $AH = \hat{H}A$ y $A^{\#}\hat{H} = HA^{\#}$. El vestimiento de Darboux de la integral de Lax-Novikov P_{2g+1} de H nos permite encontrar una integral de Lax-Novikov para $\hat{H}$ $\hat{P}_{2g+2+1} = AP_{2g+1}A^{\#}$, tal que $[\hat{P}_{2g+2+1},\hat{H}] = -2i\frac{d}{dx}f_{g+2}(\hat{u}^r(x),\partial\sigma(H)\cup\{z(r_1),z(r_1)\})$, y ${\hat{P}_{2g+2+1}}^2 = (\hat{H}-z(r_1))^2\prod_{i=0}^{2g}(\hat{H}-E_i)$. Dado que P_{2g+2+1} es dos órdenes mayor que P_{2g+1}, es posible notar que $\hat{u}(x)$ es solución estacionaria de una ecuación de un orden más alta que la de u dentro de la jerarquía de KdV. Mediante transformaciones de Crum-Darboux es posible generalizar este resultado para construir potenciales con un número arbitrario de solitones, con sus respectivas integrales de Lax-Novikov, que seran soluciones estacionarias de ecuaciones de altos órdenes en la jerarquía de KdV.

Sucesivas transformaciones de Darboux permiten obtener potenciales multisolitónicos. Para obtener potenciales reales y no singulares es necesario elegir de forma cuidadosa los estados que se utilizan en dicha transformacón, el estudio completo de la construcción de potenciales reales y no singulares en fondo de Lamé y los condensados de Bogoliubov-de Gennes asociados son estudiados en el siguiente artículo además de la supersimetria exótica asociada. El problema de construcción de Wronskianos (de autoestados) no singulares es estudiado con el fin de construir dos tipos distintos de defectos correspondientes a solitones en diferentes bandas prohibidas y así tambien distintos tipos de superpotenciales con defectos solitónicos en fondo cristalino que son soluciones del modelo de Gross y Neveu.

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Soliton defects in one-gap periodic system and exotic supersymmetry

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By applying Darboux–Crum transformations to the quantum one-gap Lamé system, we introduce an arbitrary countable number of bound states into forbidden bands. The perturbed potentials are reflectionless and contain two types of soliton defects in the periodic background. The bound states with a finite number of nodes are supported in the lower forbidden band by the periodicity defects of the potential well type, while the pulse-type bound states in the gap have an infinite number of nodes and are trapped by defects of the compression modulations nature. We investigate the exotic nonlinear $\mathcal{N}=4$ supersymmetric structure in such paired Schrödinger systems, which extends an ordinary $\mathcal{N}=2$ supersymmetry and involves two bosonic generators composed from Lax–Novikov integrals of the subsystems. One of the bosonic integrals has a nature of a central charge and allows us to liaise the obtained systems with the stationary equations of the Korteweg–de Vries and modified Korteweg–de Vries hierarchies. This exotic supersymmetry opens the way for the construction of self-consistent condensates based on the Bogoliubov–de Gennes equations and associated with them new solutions to the Gross–Neveu model. They correspond to the kink or kink-antikink defects of the crystalline background in dependence on whether the exotic supersymmetry is unbroken or spontaneously broken.

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I. INTRODUCTION

Quantum periodic finite-gap systems find many interesting applications in physics [1–22]. They can be related via the algebro-geometric approach with the integrable Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations [23,24]. The potentials of finite-gap Schrödinger systems correspond to the "snapshots" of the evolving in time generalizations of cnoidal waves solutions to the KdV equation [25]. In a similar way, via the Miura transformation, the scalar Dirac finite-gap potentials can be associated with solutions to the mKdV equation. The infinite-period limit of such potentials corresponds to reflectionless systems [26] and the solitary waves solutions to the KdV and mKdV equations.

Reflectionless second- and first-order quantum systems can be constructed via the Darboux–Crum transformations [27] from the quantum free particle Schrödinger and Dirac systems. The same transformations provide an effective dressing method for construction of Lax–Novikov integrals for these systems. The condition of conservation of them generates the higher-order nonlinear stationary equations for the KdV and mKdV hierarchies [28–31]. This picture also applies for a more general case of Zakharov–Shabat/Ablowitz–Kaup–Newell–Segur hierarchy [32].

It was shown recently in Ref. [31] that the Darboux–Crum transformations yield a possibility to relate reflectionless systems with a different number of bound states in their spectra via a soliton scattering picture. It was also demonstrated that the pairs of reflectionless Schrödinger systems are described not by the ordinary linear or nonlinear $\mathcal{N}=2$ supersymmetry, as this happens in the case of ordinary, nontransparent quantum systems related by a Darboux–Crum transformation. Instead, they are characterized by exotic nonlinear $\mathcal{N}=4$ supersymmetric structure. It is generated by two pairs of the supercharges, which are the 2×2 matrix differential operators of the odd and even orders. In addition, the exotic supersymmetric structure includes two bosonic generators composed from the Lax–Novikov integrals of subsystems, which are differential operators of higher odd order [29,30].

Among all such paired reflectionless Schrödinger systems, there is a special class, in which two lower-order supercharges have the differential order 1. In this case, one of the two bosonic integrals transmutes into the central charge of the exotic nonlinear $\mathcal{N}=4$ superalgebra, while the second bosonic integral generates rotations between the first-order and even-order supercharges. One of the first-order supercharges can be reinterpreted as the Dirac

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Hamiltonian, which is characterized by its own exotic supersymmetry associated with the central charge of the initial extended Schrödinger system. It is, in fact, the Bogoliubov–de Gennes Hamiltonian, whose potential, being the superpotential of the initial extended Schrödinger system, provides us with self-consistent condensates. The latter supply us, particularly, with kink- and kink-antikink-type solutions for the Gross–Neveu model [30]. A similar picture related to the exotic supersymmetry was also revealed in the pairs of mutually displaced one-gap Lamé systems [22].

A natural question that appears here is whether the Darboux–Crum transformations can be employed to unify the reflectionless and finite-gap properties in the same quantum system. Such a quantum system could be associated with the KdV and mKdV equations, and its potential would correspond to solitary wave solutions propagating in a background of finite-gap, cnoidal-wave-type solutions. The related question then is what happens with the exotic nonlinear supersymmetric structure in such quantum systems.

In this article, we answer the posed questions. To this aim, we apply the Darboux–Crum transformations to the quantum one-gap periodic Lamé system to introduce into its spectrum an arbitrary countable number of bound states in its two, the lowest and the intermediate, forbidden bands. This procedure will provide us the reflectionless nonperiodic one-gap potentials, which will contain two essentially different types of soliton defects in the periodic background. The nature of defects depends on the forbidden band in which they support the bound states. Coherently with this, as it will be shown, the corresponding two types of the bound states possess essentially different properties. We also investigate the exotic nonlinear supersymmetric structure associated with such quantum systems.

Some general mathematical aspects of the theory of the class of the systems we investigate here were discussed in Ref. [33]. The simplest particular examples were considered in Ref. [34]. For the discussion of the problem of defects in a more general context of integrable classical and quantum field theoretical systems, see Refs. [35–37].

The article is organized as follows. In next section, generic properties of the quantum one-gap periodic Lamé system are summarized, and its infinite-period limit corresponding to the simplest reflectionless Pöschl–Teller model with one bound state is discussed in light of Darboux–Crum transformations. In Sec. III, we consider Darboux translations for Lamé system. We apply Darboux–Crum transformations in Sec. IV to introduce soliton defects into the one-gap Lamé system. The procedure is developed first to generate an arbitrary number of periodicity defects supporting bound states in the lower forbidden band. Then, we do the same for the gap separating the allowed valence and conduction bands. As we shall see, the cases of the even and odd numbers of the bound states in the intermediate forbidden band are characterized by different Darboux–Crum schemes. Finally, we show how to generalize the construction to introduce the bound states in both forbidden bands. We discuss also the application of Darboux–Crum dressing procedure for the construction of the irreducible Lax–Novikov integrals. Section V is devoted to investigation of the exotic nonlinear $\mathcal{N}=4$ supersymmetric structure that appears in the extended Shrödinger systems composed from two arbitrary one-gap systems with periodicity defects. Special attention is given there for the most interesting from the viewpoint of physical applications case when two of the four supercharges are given by the matrix differential operators of the first order. We consider the cases of the unbroken and spontaneously broken exotic supersymmetries and indicate the relation of the obtained systems with the KdV and mKdV equations. The results are summarized in Sec. VI. We point out there further possible research directions for the development of the obtained results and some interesting applications. The Appendix is devoted to a more technical demonstration of a nonsingular nature of the constructed one-gap potentials of a generic form with an arbitrary number of the periodicity defects.

II. ONE-GAP LAMÉ SYSTEM AND ITS INFINITE-PERIOD LIMIT

In this section, we summarize generic properties of the quantum one-gap periodic Lamé system and discuss its infinite-period limit corresponding to the reflectionless Pöschl–Teller model. The Darboux transformations associate the latter system with a free particle and allow us, particularly, to identify its nontrivial Lax–Novikov integral via the dressing procedure. All this will form the basis for application of the method of the Darboux–Crum transformations to introduce two different types of nonperiodic soliton defects into the Lamé system.

A. Spectral properties of one-gap Lamé system

The quantum one-gap Lamé system is described by the Hamiltonian operator

$$H_{0,0}=-\frac{d^2}{dx^2}+V_{0,0}(x),$$
$$V_{0,0}(x)=2k^2\mathrm{sn}^2x-k^2=-2\mathrm{dn}^2x+1+k'^2, \qquad (2.1)$$

with a periodic potential $V_{0,0}(x)=V_{0,0}(x+2\mathbf{K})$.[1] The sense of the lower indices introduced here will be clarified

[1] $\mathbf{K}=\mathbf{K}(k)$ is a complete elliptic integral of the first kind corresponding to the modular parameter k, $0<k<1$. We also denote $\mathbf{K}'=\mathbf{K}(k')$, where k', $0<k'<1$, $k^2+k'^2=1$, is the complementary modular parameter. For the properties of Jacobi elliptic and related functions, see Ref. [38]. For a short summary of the properties we use here, see the Appendix in Ref. [22]. The dependence of these functions on k is not shown explicitly. In the case in which they depend on k' instead of k, we indicate such a dependence explicitly.

in what follows. The eigenstates of $H_{0,0}$ can be found in a closed analytic form for any complex eigenvalue $\mathcal{E}$. Parametrizing the latter in terms of Jacobi's elliptic dn function, $\mathcal{E}(\alpha) = \mathrm{dn}^2\alpha$, we obtain the solutions of the stationary Schrödinger equation $H_{0,0}\Psi^\alpha_\pm = \mathcal{E}(\alpha)\Psi^\alpha_\pm$,

$$\Psi^\alpha_\pm(x) = \frac{\mathrm{H}(x \pm \alpha)}{\Theta(x)} \exp\left[\mp x Z(\alpha)\right]. \tag{2.2}$$

Here, Θ, H, and Z are Jacobi's Theta, Eta, and Zeta functions, while parameter α can take arbitrary complex values. Since the periods of the doubly periodic elliptic function $\mathrm{dn}^2\alpha$ are $2\mathbf{K}$ and $2i\mathbf{K}'$, and it is an even function, without any loss of generality, one can restrict a consideration to a rectangular domain with vertices in 0, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$, and $i\mathbf{K}'$. Hamiltonian (2.1) is a Hermitian operator, and we are interested in the real eigenvalues $\mathcal{E}(\alpha)$.[2] These are provided by further restriction of the values of the parameter α to the borders of the indicated rectangle; see Fig. 1. The horizontal edges correspond to the *lower* and *upper* forbidden zones (lacunas) in the spectrum. The vertical edges correspond, respectively, to the *valence* and *conduction* bands. The necessary information on the bands' structure, including the values of quasimomentum $\kappa(\alpha)$, see below, is summarized in Table I. We supply the parameters β and γ, corresponding to real and imaginary parts of the complex parameter α, with upper index $-/+$ to distinguish whether they correspond to the lower/upper forbidden and allowed bands, respectively.

While the real parameter β^- increases in the open interval $(0, \mathbf{K})$, the energy increases in the lower, semi-infinite forbidden band but decreases in the finite gap separating the allowed bands when β^+ varies in the same interval. In the valence band, the energy increases when the parameter γ^- decreases from $\mathbf{K}'$ to 0; the variation of the parameter γ^+ in the semiopen interval $[0, \mathbf{K}')$ gives the energy monotonically increasing in the semi-infinite conduction band.

Under the shift for the real period $2\mathbf{K}$ of the potential, the eigenstates (2.2) undergo the transformation

$$\Psi^\alpha_\pm(x + 2\mathbf{K}) = \exp\left(\mp i 2\mathbf{K}\kappa(\alpha)\right)\Psi^\alpha_\pm(x),$$
$$\text{where } \kappa(\alpha) = \frac{\pi}{2\mathbf{K}} - iZ(\alpha) \tag{2.3}$$

is the *quasimomentum*, in which the first term is associated with the $2\mathbf{K}$ antiperiodicity of the Eta function, $\mathrm{H}(x + 2\mathbf{K}) = -\mathrm{H}(x)$. The analytical form of the quasimomentum $\kappa(\alpha)$ allows us to determine explicitly when it takes real or complex values and therefore to locate the allowed and forbidden bands. Thus, making use of the properties of Jacobi's Zeta function, one finds that in the lower forbidden band the quasimomentum takes pure imaginary values, $\kappa(\beta^- + i\mathbf{K}') = -i\mathrm{z}(\beta^-)$, $\mathrm{z}(\beta^-) = \frac{d}{d\beta^-}\log \mathrm{H}(\beta^-)$. In accordance with this, the quasimomentum varies in the complex plane along the imaginary axis so that $\kappa \to -i\infty$ for $\beta^- \to 0$, $\mathcal{E} \to -\infty$, and $\kappa \to 0$ when $\beta^- \to \mathbf{K}$, $\mathcal{E} \to 0$. The amplitude of the wave functions (2.2) in this band increases exponentially in one of the two directions on the real axis x, and eigenfunctions $\Psi^{\alpha=\beta^- + i\mathbf{K}'}_\pm(x)$ correspond therefore to nonphysical states. In the valence band, the quasimomentum takes real values, $\kappa(\mathbf{K} + i\gamma^-) = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^-}{\mathbf{K}'}) - \frac{d}{d\gamma^-}\log\Theta(\gamma^- + \mathbf{K}'|k')$, where it increases monotonically from $\kappa = 0$ ($\mathcal{E} = 0$) to $\kappa = \frac{\pi}{2\mathbf{K}}$ ($\mathcal{E} = k'^2$). The wave functions (2.2) inside the valence band correspond to the two linearly independent Bloch states. In the intermediate energy gap, the quasimomentum is complex valued, $\kappa(\beta^+) = \frac{\pi}{2\mathbf{K}} - iZ(\beta^+)$. In accordance with the relation $\frac{d}{d\beta}\mathrm{Z}(\beta) = \mathrm{dn}^2\beta - \frac{\mathbf{E}}{\mathbf{K}}$, where $\mathbf{E}$ is the complete elliptic integral of the second kind, and $k'^2 < \frac{\mathbf{E}}{\mathbf{K}} < 1$, the imaginary part in $\kappa(\beta^+)$ varies monotonically in the interval $\beta^+ \in (0, \beta_*]$, $0 < \mathrm{Z} \le \mathrm{Z}(\beta_*)$, where β_* corresponds to the equality $\mathrm{dn}^2\beta_* = \frac{\mathbf{E}}{\mathbf{K}}$ and then decreases monotonically approaching the zero value in the interval $\beta^+ \in (\beta_*, \mathbf{K})$. In the conduction band, like in the valence band, the quasimomentum takes real values, $\kappa(i\gamma^+) = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^+}{\mathbf{K}'}) - \frac{d}{d\gamma^+}\log\mathrm{H}(\gamma^+ + \mathbf{K}'|k')$. It increases here monotonically from $\frac{\pi}{2\mathbf{K}}$ ($\mathcal{E} = 1$) to $+\infty$ ($\mathcal{E} \to \infty$). Inside this band, for any value of the energy, the two wave functions (2.2) correspond to the two linearly independent physical Bloch states.

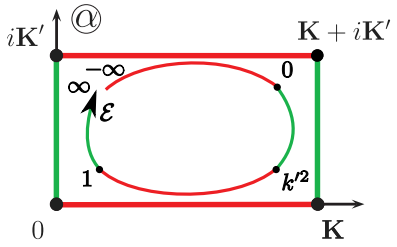

FIG. 1 (color online). Spectrum of the one-gap Lamé system (2.1) as a function of complex parameter α.

The properties of a periodic quantum system are effectively reflected by the discriminant $\mathcal{D}(\mathcal{E})$ (Lyapunov function) of the corresponding stationary Schrödinger equation, which is defined as a trace of the monodromy matrix representing the operator of the translation for the period of the potential [23,41–43]. Its form $\mathcal{D}(\mathcal{E}) = 2\cos\left(2\mathbf{K}\kappa(\mathcal{E})\right)$ for the one-gap Lamé system (2.1) is shown on Fig. 2. In the lower prohibited zone and in the valence band, the explicit analytic form is given, respectively, by $\mathcal{D}(\mathcal{E}(\beta^- + i\mathbf{K}')) = 2\cosh\left(2\mathbf{K}\mathrm{z}(\beta^-)\right)$ and $\mathcal{D}(\mathcal{E}(\mathbf{K} + i\gamma^-)) = 2\cos\left(2\mathbf{K}\kappa(\gamma^-|k')\right)$. In the energy gap separating the valence and conduction bands, it reduces to

[2]The PT-symmetric generalization [39,40] of (2.1) can also be associated with real values of $\mathcal{E}(\alpha)$; see below.

TABLE I. Bands and their characteristics. Here $z(\beta^-) = Z(\beta^-) + \mathrm{cn}\,\beta^- \mathrm{ds}\,\beta^-$, $\kappa(\gamma^\pm|k') = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^\pm}{\mathbf{K}}) - Z(\gamma^\pm|k') + f_\pm$, $f_- = k'^2 \mathrm{sn}(\gamma^-|k')\mathrm{cd}(\gamma^-|k')$, and $f_+ = \mathrm{sn}(\gamma^+|k')\mathrm{dc}(\gamma^+|k')$.

Band	$\alpha = \beta + i\gamma$	$\mathcal{E}(\alpha)$	$\kappa(\alpha)$
Lower forbidden	$\beta \equiv \beta^- \in (0, \mathbf{K})$, $\gamma = \mathbf{K}'$	$(-\infty, 0) \ni \mathcal{E} = -\mathrm{cs}^2\beta^-$	$-iz(\beta^-)$
Valence	$\beta = \mathbf{K}$, $\gamma \equiv \gamma^- \in [0, \mathbf{K}']$	$[0, k'^2] \ni \mathcal{E} = k'^2\mathrm{cd}^2(\gamma^-\|k')$	$\kappa(\gamma^-\|k')$
Upper forbidden (gap)	$\beta \equiv \beta^+ \in (0, \mathbf{K})$, $\gamma = 0$	$(k'^2, 1) \ni \mathcal{E} = \mathrm{dn}^2\beta^+$	$\frac{\pi}{2\mathbf{K}} - iZ(\beta^+)$
Conduction	$\beta = 0$, $\gamma \equiv \gamma^+ \in [0, \mathbf{K}')$	$[0, +\infty) \ni \mathcal{E} = \mathrm{dc}^2(\gamma^+\|k')$	$\kappa(\gamma^+\|k')$

$\mathcal{D}(\mathcal{E}(\beta^+)) = -2\cosh(2\mathbf{K}Z(\beta^+))$. The minimum of the curve at $\mathcal{E} = \mathrm{dn}^2\beta_* = \frac{\mathbf{E}}{\mathbf{K}}$ corresponds to the maximum value $Z(\beta_*) > 0$ of the Zeta function. In the conduction band, we have $\mathcal{D}(\mathcal{E}(\gamma^+)) = 2\cos(2\mathbf{K}\kappa(\gamma^+|k'))$. The infinite number of oscillations of the curve between -2 and $+2$ extrema values of the $\mathcal{D}(\mathcal{E})$ is associated in this band with the zero of $\mathrm{cn}(\gamma^+|k')$ at $\gamma^+ = \mathbf{K}'$ appearing in the denominator of the function f_+ in the structure of $\kappa(\gamma^+|k')$; see Table I.

At the edges of the valence and conduction bands, where $|\mathcal{D}| = 2$, $\frac{d\mathcal{D}}{d\mathcal{E}} \neq 0$, the two wave functions (2.2) reduce, up to numerical factors, to the same periodic, $\psi_1 = \mathrm{dn}\,x$ ($\mathcal{E} = 0$), and antiperiodic, $\psi_2 = \mathrm{cn}\,x$ ($\mathcal{E} = k'^2$) and $\psi_3 = \mathrm{sn}\,x$ ($\mathcal{E} = 1$), eigenstates. The second, linear independent eigenfunctions at the edges of the valence and conduction bands are given by $\Psi_i(x) = \psi_i(x)\mathcal{I}_i$, $i = 1, 2, 3$, where $\mathcal{I}_i(x) = \int dx/\psi_i^2(x)$ are expressed in terms of the incomplete elliptic integral of the second kind, $\mathrm{E}(x) = \int_0^x \mathrm{dn}^2 x dx$: $\mathcal{I}_1(x) = \frac{1}{k'^2}\mathrm{E}(x + \mathbf{K})$, $\mathcal{I}_2(x) = x - \frac{1}{k'^2}\mathrm{E}(x + \mathbf{K} + i\mathbf{K}')$, $\mathcal{I}_3(x) = x - \mathrm{E}(x + i\mathbf{K}')$. The functions $\Psi_i(x)$ are not bounded on the real line and correspond to nonphysical eigenstates of the Lamé Hamiltonian operator. They also can be obtained from the states (2.2) by differentiation in α. Namely, derivatives of the functions $\Psi_+^\alpha(x)$ in α at $\alpha = 0$ and $\alpha = \mathbf{K}$ give some linear combinations of the functions $\psi_i(x)$ and $\Psi_i(x)$ with $i = 3$ and $i = 2$, respectively, while the derivative of the function (2.6) in parameter β^- at $\beta^- = \mathbf{K}$ gives a linear combination of $\psi_1(x)$ and $\Psi_1(x)$.

For any value of the parameter α, under the parity reflection, $Pf(x) = f(-x)$, the states (2.2) satisfy the relation

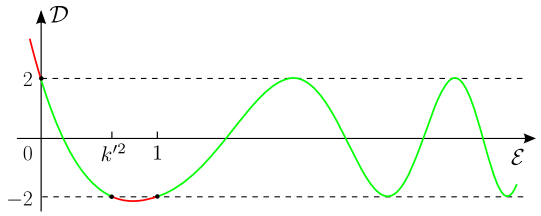

FIG. 2 (color online). The discriminant $\mathcal{D}(\mathcal{E})$ of the one-gap Lamé system. The scale is linear in energy for $\mathcal{E} < 1$, while for $\mathcal{E} > 1$ a logarithmic scale is used here. The parts shown in red correspond to the lower ($\mathcal{E} < 0$) and to the upper ($k'^2 < \mathcal{E} < 1$) forbidden bands.

$$P\Psi_\pm^\alpha(x) = -\Psi_\mp^\alpha(x). \tag{2.4}$$

The properties of the wave functions (2.2) in corresponding bands under the T, $Tf(x) = f^*(x)$, and the composed PT operations [39,40] are shown in Table II.

Notice that in the lower forbidden band

$$\Psi_\pm^{\beta^- + i\mathbf{K}'}(x) = \pm iq^{-1/4}\exp\left(-i\frac{\pi\beta^-}{2\mathbf{K}}\right)F(\pm x; \beta^-), \tag{2.5}$$

where

$$F(x; \beta^-) = \frac{\Theta(x + \beta^-)}{\Theta(x)}\exp(-xz(\beta^-)) \tag{2.6}$$

is a real-valued function of x, which takes positive values, $F(x; \beta^-) > 0$. Here, $q = \exp(-\pi\mathbf{K}'/\mathbf{K})$ is Jacobi's nome, and we used the relation $\mathrm{H}(x + i\mathbf{K}') = iq^{-1/4}\exp(-i\frac{\pi x}{2\mathbf{K}})\Theta(x)$. In this band, one can employ alternatively the real functions $F(x; \beta^-)$ and $F(-x; \beta^-) = PF(x; \beta^-)$ as two linear independent solutions.

The operator PT distinguishes whether the function (2.2) belongs to the forbidden or allowed band. When it corresponds to the physical Bloch state, it is also the eigenfunction of the PT. In contrast, the functions (2.2) from the forbidden bands cease to be eigenstates of the PT operator. Instead, certain linear combinations of the two states (2.2) with the opposite sign of the quasimomentum have to be taken to create the eigenstates of the PT operator in the forbidden bands.

B. Infinite period limit: Reflectionless Pöschl–Teller system and Darboux transformations

Before we pass to the discussion of the introduction of the periodicity defects, corresponding to solitons, into the

TABLE II. Properties of the eigenfunctions under the T and PT operations. Here, $c = \exp(i\frac{\pi\beta^-}{\mathbf{K}})$.

Band	$\Psi_\pm^\alpha(x)$	$T\Psi_\pm^\alpha(x)$	$PT\Psi_\pm^\alpha(x)$
Lower forbidden	$\Psi_\pm^{\beta^- + i\mathbf{K}'}(x)$	$-c\Psi_\pm^{\beta^- + i\mathbf{K}'}(x)$	$c\Psi_\mp^{\beta^- + i\mathbf{K}'}(x)$
Valence	$\Psi_\pm^{\mathbf{K} + i\gamma^-}(x)$	$-\Psi_\mp^{\mathbf{K} + i\gamma^-}(x)$	$\Psi_\pm^{\mathbf{K} + i\gamma^-}(x)$
Upper forbidden (gap)	$\Psi_\pm^{\beta^+}(x)$	$\Psi_\pm^{\beta^+}(x)$	$-\Psi_\mp^{\beta^+}(x)$
Conduction	$\Psi_\pm^{i\gamma^+}(x)$	$\Psi_\mp^{i\gamma^+}(x)$	$-\Psi_\pm^{i\gamma^+}(x)$

spectrum of the one-gap Lamé system, we consider briefly the analogous procedure for the infinite-period limit case. The picture in such a limit case is more simple and transparent, and it is useful to bear it in mind when we generalize the method to the very Lamé system.

In the infinite-period limit $\mathbf{K} \to \infty$, which is equivalent to any of the three limits $k \to 1$, $k' \to 0$, or $\mathbf{K}' \to \pi/2$, operator (2.1) transforms into the Hamiltonian of the reflectionless Pöschl–Teller system

$$H_1 = -\frac{d^2}{dx^2} + V_1(x), \qquad V_1(x) = -\frac{2}{\cosh^2 x} + 1. \tag{2.7}$$

In this limit, the valence band shrinks into one discrete energy level $\mathcal{E} = 0$. The wave functions (2.2) of the valence band with $\alpha = \mathbf{K} + i\gamma^-$, $\gamma^- \in [0, \mathbf{K}']$ transform into the unique bound state described by the normalizable wave function $\Psi_{\mathcal{E}=0}(x) = \mathrm{sech} x$. The conduction band, parametrized by $\alpha = i\gamma^+$, $\gamma^+ \in [0, \mathbf{K}')$, transforms into the scattering part of the spectrum of the system (2.7). In the limit, we have $\gamma^+ \in [0, \frac{\pi}{2})$. Introducing the notation $\tan\gamma^+ = \mathrm{k}$, $0 \le \mathrm{k} < \infty$, we find that the rescaled wave functions $q^{-1/4}\Psi_{\mp}^{\alpha=i\gamma^+}(x)$ of the conduction band transform, up to an inessential constant multiplicative factor, into the wave functions

$$\Psi_{\pm}^{\mathrm{k}}(x) = (\pm i\mathrm{k} - \tanh x)e^{\pm i\mathrm{k}x}. \tag{2.8}$$

Corresponding energy $\mathcal{E} = \mathrm{dn}^2(i\gamma^+|k) = \mathrm{dc}^2(\gamma^+|k')$ transforms in the limit $k' \to 0$ into $1/\cos^2\gamma^+ = 1 + \mathrm{k}^2$, which is the eigenvalue of the eigenstates (2.8) of the Pöschl–Teller Hamiltonian (2.7). The nondegenerate state $\Psi^0 = \tanh x$ ($\mathrm{k} = 0$) corresponds here to the state of energy $\mathcal{E} = 1$ described by $\mathrm{sn}x$ at the edge of the conduction band of the Lamé system (2.1).

The scattering states (2.8) can be presented in the form $\Psi_{\pm}^{\mathrm{k}}(x) = A_\varphi e^{\pm i\mathrm{k}x}$ in terms of the first-order differential operator

$$A_\varphi = \varphi(x)\frac{d}{dx}\frac{1}{\varphi(x)} = \frac{d}{dx} - \tanh x, \qquad \varphi(x) = \cosh x. \tag{2.9}$$

Operator A_φ together with the Hermitian conjugate $A_\varphi^\dagger$ intertwine the reflectionless system (2.7) with the free particle Hamiltonian shifted for an additive constant,

$$H_0 = -\frac{d^2}{dx^2} + 1, \tag{2.10}$$

and provide the factorization of both:

$$A_\varphi A_\varphi^\dagger = H_1, \qquad A_\varphi^\dagger A_\varphi = H_0,$$
$$A_\varphi H_0 = H_1 A_\varphi, \qquad A_\varphi^\dagger H_1 = H_0 A_\varphi^\dagger. \tag{2.11}$$

Relations (2.11) correspond to the Darboux transformations that relate the free particle system with the reflectionless Pöschl–Teller system. The alternative form to express the same relation between the systems corresponds to the equality

$$H_1 = H_0 - 2\frac{d^2}{dx^2}\log\varphi(x). \tag{2.12}$$

The wave function $\varphi(x) = \cosh x$ is a nodeless nonphysical eigenstate of the free particle H_0, and the operator A_φ produces an almost isospectral mapping of all the physical and nonphysical states of H_0, except $\varphi(x)$, $A_\varphi\varphi(x) = 0$, into corresponding states of the system H_1. The only physical bound state $\Psi_{\mathcal{E}=0}(x) = \mathrm{sech} x$ of H_1 of zero energy, for which there is no bound state analog in the physical spectrum of H_0, is obtained by applying the operator A_φ to the wave function $\tilde{\varphi}(x) = \varphi(x)\int\frac{dx}{\varphi^2(x)}$. This is the nonphysical eigenstate of (2.10) of the same zero eigenvalue as $\varphi(x)$. It reduces here just to the derivative of the latter, $\tilde{\varphi}(x) = \sinh x = \varphi'(x)$. Analogously, the application of the operator $A_\varphi^\dagger$ to the eigenstates of H_1 in correspondence with the last relation in (2.11) produces the eigenstates of H_0. The unique bound state $\Psi_{\mathcal{E}=0}(x) = \mathrm{sech} x$ of H_1 is the zero mode of the first-order operator $A_\varphi^\dagger$.

The free particle system (2.10) has a nontrivial integral $p = -i\frac{d}{dx}$. It distinguishes the plane waves $e^{\pm i\mathrm{k}x}$, which are the eigenstates of H_0 of the same energy, and detects a unique nondegenerate state $\Psi_{\mathcal{E}=1}(x) = 1$ corresponding to $\mathrm{k} = 0$ by annihilating it. In correspondence with the last two relations in (2.11) and the described picture of the mapping associated with the Darboux transformations, one finds that the operator

$$\mathcal{P} = -iA_\varphi\frac{d}{dx}A_\varphi^\dagger \tag{2.13}$$

is the Hermitian integral for the reflectionless system H_1. We refer to this as the dressing procedure. Similarly to p, this operator distinguishes the eigenstates (2.8), being analogs of the plane wave states for the free particle, $\mathcal{P}\Psi_{\pm}^{\mathrm{k}}(x) = \pm\mathrm{k}(1+\mathrm{k}^2)\Psi_{\pm}^{\mathrm{k}}(x)$. It annihilates the lowest nondegenerate state $\Psi^0(x) = \tanh x$ in the scattering sector, and the bound state[3] $\Psi_{\mathcal{E}=0}(x) = \mathrm{sech} x$. Integral (2.13) satisfies the Burchnall–Chaundy relation [45]

$$\mathcal{P}^2 = H_1^2(H_1 - 1). \tag{2.14}$$

Since the free particle has the integral $p = -i\frac{d}{dx}$, the H_0 and the Pöschl–Teller Hamiltonian (2.7) can be intertwined

[3]Being the third-order differential operator, (2.13) also turns into zero the state $\varphi(x) = \cosh x$, which is a nonphysical eigenstate of the free particle Hamiltonian (2.10) [44].

not only by the first-order operator (2.9) and its conjugate $A_\varphi^\dagger$ but also by the second-order operators

$$B_\varphi = A_\varphi \frac{d}{dx} \quad \text{and} \quad B_\varphi^\dagger. \tag{2.15}$$

The first- and second-order intertwining operators together with the integrals p and $\mathcal{P}$ of the systems H_0 and H_1 constitute the building blocks of the exotic centrally extended $\mathcal{N}=4$ nonlinear supersymmetry of the system described by the 2×2 matrix Hamiltonian $\mathcal{H}=\mathrm{diag}(H_0,H_1)$ [31].

Suppose now that we want to construct another reflectionless system proceeding from the Pöschl–Teller system (2.7) by means of a new Darboux transformation, or a composition of them, that corresponds to the Darboux–Crum transformation. There are three different ways to do this. First, one can construct a reflectionless system with an additional, second bound state lying below the unique, zero energy bound state of the system (2.7). Another case corresponds to the situation in which we want to introduce a bound state with the energy level lying between the zero energy level of the already existing bound state and the edge of the scattering sector of energy $\mathcal{E}=1$. At last, one can construct a reflectionless system completely isospectral to the system (2.7) but with the displaced potential ("soliton center"). Having at hands the building blocks corresponding to the described three possibilities, by the appropriate generalization of the procedure, we can construct a reflectionless system with an arbitrary number of bound states and arbitrary positions of the corresponding soliton centers [29,30].

The first situation is realized by the construction in a way similar to (2.9) of the Darboux generator on the basis of the nodeless function

$$\varphi_1(x;\kappa_1,\tau_1) = A_\varphi \sinh\kappa_1(x+\tau_1), \tag{2.16}$$

where $\kappa_1>1$ and τ_1 is an arbitrary real parameter. The function $\varphi_1(x;\kappa_1,\tau_1)$ is the nonphysical eigenstate of (2.7) with energy $1-\kappa_1^2$, and τ_1 is associated with the center (phase) of the second soliton (the first soliton is characterized by $\tau_0=0$ and the amplitude $\kappa_0=1$) in the potential of the system

$$H_2 = H_1 - 2\frac{d^2}{dx^2}\log\varphi_1(x) \tag{2.17}$$

with two bound states; cf. (2.12). Note that alternatively H_2 can be presented in terms of the second-order Darboux–Crum transformation applied to the free particle, $H_2 = H_0 - 2\frac{d^2}{dx^2}\log\mathbb{W}(x)$, where $\mathbb{W}(x)$ is the Wronskian of the two nonphysical states of the free particle, $\varphi=\cosh x$ and $\phi=\sinh\kappa_1(x+\tau_1)$, $\mathbb{W}(x)=W(\varphi,\phi)=\varphi\phi'-\varphi'\phi$.

To obtain a reflectionless system with an additional bound state inside the energy interval (0, 1), which separates the bound state level of the system (2.7) with the continuous part of the spectrum, one can apply to (2.7) the Darboux–Crum transformation generated by the two nonphysical states $\phi_1(x;\kappa_1,\tau_1)=A_\varphi\cosh\kappa_1(x+\tau_1)$ and $\phi_2(x;\kappa_2,\tau_2)=A_\varphi\sinh\kappa_2(x+\tau_2)$. If we restrict the parameters $\kappa_{1,2}$ by the condition $0<\kappa_1<\kappa_2<1$, the corresponding Wronskian $\mathbb{W}(x)=W(\phi_1,\phi_2)$ has no zeros. This produces a system with a regular reflectionless potential

$$V_3(x) = V_1(x) - 2\frac{d^2}{dx^2}\log\mathbb{W}(x), \tag{2.18}$$

which has three bound states with energies $1-\kappa_1^2$, $1-\kappa_2^2$, and 0. Sending then one of the two translation parameters, τ_2 or τ_1, to any of the limits $+\infty$ or $-\infty$, we get a reflectionless system with two bound states of energies $1-\kappa_1^2$ and 0 when we send $|\tau_2|\to\infty$, or with energies $1-\kappa_2^2$ and 0 when $|\tau_1|\to\infty$. The indicated limit changes the translation parameters of the remaining added soliton as well as of the initial one with $\kappa_0=1$ and $\tau_0=0$ in correspondence with the picture of soliton scattering; see Ref. [31].

There is another possibility to introduce one additional bound state into the spectrum of the system (2.7) with the energy inside the interval (0, 1). One can apply to (2.7) a Darboux transformation constructed on the basis of its nonphysical state $\phi(x;\kappa,\tau)=A_\varphi\sinh\kappa(x+\tau)$, $0<\kappa<1$. This will produce a singular system. Shifting then $\tau\to\tau+i\frac{\pi}{2\kappa}(1-\kappa)$ and $x\to x+i\frac{\pi}{2}$, we get a regular reflectionless system with two bound states with energies $1-\kappa^2$ and 0.

Finally, to produce a system completely isospectral to the system (2.7), one can apply to the latter the Darboux transformation based on the function [31] $f(x;\kappa)=A_\varphi\exp(\kappa x)$, where $\kappa>1$. In the present simplest case of H_1, this will give us the shifted system (2.7), in which the argument of the potential x changes for[4] $x+\lambda$, where $\lambda=\frac{1}{2}\log\frac{\kappa-1}{\kappa+1}$.

In all three indicated cases, the corresponding extended system $\mathcal{H}=\mathrm{diag}(H_1,\tilde{H})$ will be described by the exotic centrally extended nonlinear $\mathcal{N}=4$ supersymmetry [29–31]. Such reflectionless systems will correspond to the $k\to 1$ limit of the systems obtained from the one-gap Lamé system by introducing into it the periodicity defects by means of the appropriate Darboux(–Crum) transformation.

[4]In the case of a reflectionless system with $n>1$ bound states, the isospectral deformation of the potential, which can be generated by applying the appropriate Darboux–Crum transformation, corresponds to a "snapshot" of the evolved n-soliton solution of the Korteweg–de Vries equation; see Refs. [29–31]. In that case, like in the case of Lamé system with periodicity defects we consider below, the form of the isospectrally deformed potential is different from the original one.

In the subsequent sections, we describe how to introduce such periodicity defects and discuss the associated exotic nonlinear supersymmetric structure.

III. DARBOUX TRANSLATIONS OF THE LAMÉ SYSTEM

Assume that we have a system described by a Hamiltonian operator of the most general form $H = -\frac{d^2}{dx^2} + U(x)$ and that $\psi(x)$ is its an arbitrary physical, or nonphysical eigenstate, $H\psi = \mathcal{E}\psi$. As in (2.9), we define the first-order operators

$$A_\psi = \psi \frac{d}{dx}\frac{1}{\psi} = \frac{d}{dx} + \Delta(x), \qquad \Delta(x) = -\frac{d}{dx}\log\psi(x), \tag{3.1}$$

and

$$A_\psi^\sharp = -\frac{1}{\psi}\frac{d}{dx}\psi = -\frac{d}{dx} + \Delta(x). \tag{3.2}$$

If $\psi(x)$ is a real valued function modulo a possible complex multiplicative constant, then the operators A_ψ and $A_\psi^\sharp$ are mutually conjugate, $A_\psi^\sharp = A_\psi^\dagger$. Another, linear independent eigenstate of H of the same eigenvalue $\mathcal{E}$ is given by $\tilde{\psi}(x) = \psi(x)\int dx/\psi^2(x)$. The action of the operator A_ψ on this eigenstate produces a kernel of the operator $A_\psi^\sharp$, $A_\psi\tilde{\psi}(x) = 1/\psi(x)$. The second-order operator $A_\psi^\sharp A_\psi = -\frac{d^2}{dx^2} + \Delta^2(x) - \Delta'(x)$ has exactly the same kernel, spanned by $\psi(x)$ and $\tilde{\psi}(x)$, as the second-order differential operator $H - \mathcal{E}$, and therefore $A_\psi^\sharp A_\psi = H - \mathcal{E}$, and $\Delta^2(x) - \Delta'(x) = U(x) - \mathcal{E}$.

Consider now the operator $A_\psi A_\psi^\sharp = -\frac{d^2}{dx^2} + \Delta^2(x) + \Delta'(x) = A_\psi^\sharp A_\psi + 2\Delta'(x) \equiv \tilde{H} - \mathcal{E}$. The wave function $1/\psi(x)$ is the eigenstate of the Schrödinger Hamiltonian operator $\tilde{H}$ of eigenvalue $\mathcal{E}$. Another, linear independent eigenstate of $\tilde{H}$ of the same eigenvalue $\mathcal{E}$ is $\frac{1}{\psi(x)}\int \psi^2(x)dx$. The latter is mapped by the operator $A_\psi^\sharp$ into the state $\psi(x)$ being the zero mode of A_ψ.

Let us return now to the Lamé system (2.1). Its eigenstates $\Psi_+^\alpha(x)$ obey the property

$$\Psi_+^\alpha(-x - \alpha - i\mathbf{K}') = -\Psi_-^\alpha(x + \alpha + i\mathbf{K}') = \frac{\mathcal{C}(\alpha)}{\Psi_+^\alpha(x)}, \tag{3.3}$$

where $\mathcal{C}(\alpha) = -\exp\left(\alpha(\mathrm{Z}(\alpha) + i\frac{\pi}{2\mathbf{K}}) + i\mathbf{K}'\mathrm{Z}(\alpha)\right)$. Taking $\psi(x) = \Psi_+^\alpha(x)$ in (3.1), we obtain the factorization for the one-gap Lamé Hamiltonian,

$$A_{\Psi_+^\alpha}^\sharp A_{\Psi_+^\alpha} = H_{0,0}(x) - \mathcal{E}(\alpha). \tag{3.4}$$

Making use of the relation (3.3), we find then that

$$A_{\Psi_+^\alpha} A_{\Psi_+^\alpha}^\sharp = H_{0,0}(x + \alpha + i\mathbf{K}') - \mathcal{E}(\alpha). \tag{3.5}$$

As the Darboux-partner of the Lamé Hamiltonian $H_{0,0}(x)$, we obtain therefore the translated Hamiltonian operator $H_{0,0}(x + \alpha + i\mathbf{K}')$.

In the case of the lower prohibited band, the wave function $\Psi_+^{\beta^- + i\mathbf{K}'}(x)$ reduces to the real function $F(x;\beta)$ modulo a constant multiplier, see Eqs. (2.5) and (2.6), and we have $A_{\Psi_+^\alpha} = A_F$, $A_{\Psi_+^\alpha}^\sharp = A_F^\dagger$. The property $\mathrm{dn}(x + 2i\mathbf{K}') = -\mathrm{dn}x$ gives us then in (3.5) the same Hermitian Lamé Hamiltonian operator but shifted for the real distance β^-, $0 < \beta^- < \mathbf{K}$, $H_{0,0}(x + \alpha + i\mathbf{K}') = H_{0,0}(x + \beta^-)$. The obtained Darboux transformations, supersymmetry, and physics associated with them were studied in diverse aspects in Ref. [22]. Note here that the real function $F(x;\beta^-)$ takes positive values for all x, blows up exponentially when $x \to -\infty$, and tends to zero for $x \to +\infty$. The limit case $\beta^- = \mathbf{K}$ corresponds to a translation for the half of the period of Lamé Hamiltonian. It is produced on the basis of the ground state $\psi(x) = \mathrm{dn}x$ [19]. The obtained Darboux transformations are analogous to the translation transformations in the case of the Pöschl–Teller system (2.7) with one bound state, which are constructed on the basis of the exponentlike nonphysical eigenstates $\psi = A_\varphi \exp\kappa x$, $\kappa > 1$, of H_1.

In the forbidden band separating the allowed bands, the eigenfunction $\Psi_+^{\beta^+}(x)$ takes real values, but it has an infinite number of zeros at the points $-\beta^+ + 2n\mathbf{K}$, $n \in \mathbb{Z}$. In this case, relation (3.4) gives us the factorization of the Lamé Hamiltonian $H_{0,0}(x)$ in terms of the singular mutually conjugate Darboux generators. The alternative product (3.5) of these first-order differential operators produces the Hermitian operator $H_{0,0}(x + \beta^+ + i\mathbf{K}')$ with the singular Treibich–Verdier potential [46]

$$V_{0,0}(x + \beta^+ + i\mathbf{K}') = \frac{2}{\mathrm{sn}^2(x + \beta^+)} - k^2, \tag{3.6}$$

where we have taken into account the identity $\mathrm{sn}(x + i\mathbf{K}') = 1/k\,\mathrm{sn}\,x$. The limiting case $\beta^+ = 0$ corresponds to the singular Darboux transformation constructed on the basis of the eigenfunction $\psi(x) = \mathrm{sn}x$ at the edge of the conduction band. Another limit case $\beta^+ = \mathbf{K}$ gives rise to the singular transformation based on the eigenfunction $\psi(x) = \mathrm{cn}x$ at the edge of the valence band, for which the Treibich–Verdier potential reduces to

$$V_{0,0}(x + \mathbf{K} + i\mathbf{K}') = 2\mathrm{dc}^2 x - k^2, \tag{3.7}$$

where we have employed the identity $\mathrm{sn}(x + \mathbf{K} + i\mathbf{K}') = \mathrm{dn}x/k\mathrm{cn}x$.

Inside the valence band, the eigenstate $\Psi_+^{\mathbf{K}+i\gamma^-}(x)$ takes nonzero but complex values. The Darboux partner (3.5) reduces in this case to the nonsingular PT-symmetric Hamiltonian with the potential

$$V_{0,0}(x+\alpha+i\mathbf{K}') = 2\mathrm{dc}^2(x+i\gamma^-) - k^2. \tag{3.8}$$

The edge value $\gamma^- = \mathbf{K}'$ corresponds here to the regular Hermitian Lamé Hamiltonian operator shifted for the half-period, $H_{0,0}(x+\mathbf{K})$. Another edge value $\gamma^- = 0$ gives the singular Hermitian Treibich–Verdier Hamiltonian (3.7) obtained on the basis of the edge state $\psi(x) = \mathrm{cn}x$.

At last, inside the conduction band, the Hamiltonian in (3.5) reduces to the regular PT-symmetric operator with the potential

$$V_{0,0}(x+i\gamma^+ + i\mathbf{K}') = \frac{2}{\mathrm{sn}^2(x+i\gamma^+)} - k^2. \tag{3.9}$$

The edge case $\gamma^+ = 0$ reduces to the singular Treibich–Verdier potential generated via the choice $\psi(x) = \mathrm{sn}x$.

The described first-order Darboux transformations can also be considered for the values of the parameter α lying inside the rectangle in Fig. 1. In this case, the partner Hamiltonian will be nonsingular with the potential taking complex values, which, however, will be neither a Hermitian nor PT-symmetric operator. Indeed, under Hermitian conjugation, the shifted Hamiltonian operator from (3.5) transforms as $(H_{0,0}(x+\alpha+i\mathbf{K}'))^\dagger = H_{0,0}(x+\alpha^*+i\mathbf{K}')$, where we have taken into account the pure imaginary period $2i\mathbf{K}'$ of the potential $V_{0,0}(x)$. Analogously, we have $PT(H_{0,0}(x+\alpha+i\mathbf{K}')) = H_{0,0}(x-\alpha^*+i\mathbf{K}')$, where the even nature of the potential has additionally been taken into account. The shifted Hamiltonian is therefore Hermitian if $\alpha-\alpha^* = 2n\mathbf{K}+2im\mathbf{K}'$, $n,m\in\mathbb{Z}$, while it is PT symmetric when $\alpha+\alpha^* = 2n\mathbf{K}+2im\mathbf{K}'$. For the α region shown in Fig. 1, the first condition is satisfied only on the upper and lower horizontal edges of the rectangle, which correspond to the prohibited zones in the spectrum, while the second relation takes place only on the vertical edges corresponding to the allowed valence and conduction bands.

Below, we shall see that the higher-order Darboux–Crum transformation corresponding to a composition of the Darboux transformations, each of which generates the translated Lamé system of the form (3.5), produces the Lamé system with a shift of the argument equal to the sum of individual translations.

IV. LAMÉ SYSTEM DEFORMED BY NONPERIODIC, SOLITON DEFECTS

In this section, we show how to introduce the reflectionless, soliton (nonperiodic) defects into the one-gap Lamé system.

A. Lower forbidden band

The real-valued eigenfunction $F(x;\beta^-)$ in the lower prohibited band has the modulated exponentlike behavior. Let us take a linear combination of the two eigenfunctions of the same eigenvalue,

$$\mathcal{F}_\pm(x;\beta^-,C) = CF(x;\beta^-) \pm \frac{1}{C}F(-x;\beta^-), \tag{4.1}$$

where $\mathbf{K} > \beta^- > 0$ and a real parameter C is restricted by the condition $C > 0$. These states have the properties $\mathcal{F}_\pm(-x;\beta^-,C^{-1}) = \pm\mathcal{F}_\pm(x;\beta^-,C)$. The function $\mathcal{F}_+(x;\beta^-,C)$ takes strictly positive values and blows up exponentially in the limits $x\to\pm\infty$. The function $\mathcal{F}_-(x;\beta^-,C)$, on the other hand, tends exponentially to $+\infty$ and $-\infty$ when x tends to $-\infty$ and $+\infty$, respectively, and has a unique zero whose position depends on the values of the parameters β^- and C. The form of the functions $\mathcal{F}_\pm(x;\beta^-,C)$ is shown in Fig. 3.

Construct now the first-order operator

$$A_{0,1} = \mathcal{F}_+(1)\frac{d}{dx}\frac{1}{\mathcal{F}_+(1)} = \frac{d}{dx} - \frac{d}{dx}\log\mathcal{F}_+(1), \tag{4.2}$$

where $\mathcal{F}_+(1) = \mathcal{F}_+(x;\beta_1^-,C_1)$. We have $A_{0,1}^\dagger A_{0,1} = H_{0,0}-\varepsilon_1^-$, and $A_{0,1}A_{0,1}^\dagger = H_{0,1}-\varepsilon_1^-$, where $\varepsilon_1^- \equiv \mathcal{E}(\beta_1^- + i\mathbf{K}') = -\mathrm{cn}^2\beta_1^-/\mathrm{sn}^2\beta_1^- < 0$,

$$H_{0,1} = H_{0,0} - 2\frac{d^2}{dx^2}(\log\mathcal{F}_+(1)) = -\frac{d^2}{dx^2} + V_{0,1}(x), \tag{4.3}$$

$$V_{0,1}(x) = 1 + k'^2 - 2\frac{\mathbf{E}}{\mathbf{K}} - 2\frac{d^2}{dx^2}(\log\chi_{0,1}^{\beta_1^-}(x;C_1)), \tag{4.4}$$

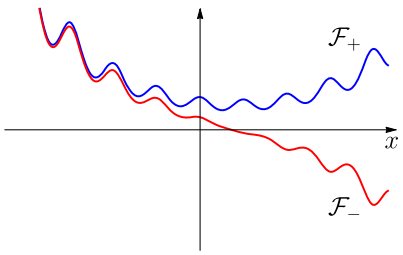

FIG. 3 (color online). At $C=1$, $\mathcal{F}_+(x;\beta^-,C)$ is an even function, while $\mathcal{F}_-(x;\beta^-,C)$ is odd. The symmetry of nonphysical eigenfunctions $\mathcal{F}_\pm(x;\beta^-,C)$ of $H_{0,0}$ is broken for $C\neq 1$. Here, the case $C>1$ is shown. With C increasing, the minimum of $\mathcal{F}_+(x;\beta^-,C)>0$ and zero of $\mathcal{F}_-(x;\beta^-,C)$ are displaced to the right. A similar situation occurs when $0<C<1$ but with a displacement to the negative coordinate axis. In fact, the form of the functions for $0<C<1$ is obtained from that for $C>1$ via the relation $\mathcal{F}_\pm(x;\beta^-,C) = \pm\mathcal{F}_\pm(-x;\beta^-,C^{-1})$.

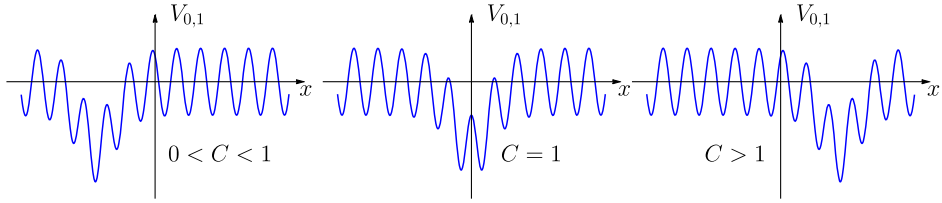

FIG. 4 (color online). Potential with a one-soliton defect that supports a bound state in the lower forbidden band. The soliton is broader when the energy of the bound state is closer to zero, and a greater number of oscillations are observable within it. The depth (amplitude) of the soliton, on the other hand, increases when the negative energy of the bound state is deeper. The sequence of the pictures illustrates the propagation of the soliton in the periodic background of the Lamé potential.

$$\chi_{0,1}^{\beta_1^-}(x;C_1) = C_1\Theta(x+\beta_1^-)\exp(-xz(\beta_1^-)) + \frac{1}{C_1}\Theta(x-\beta_1^-)\exp(xz(\beta_1^-)). \tag{4.5}$$

The $\Theta(x)$ function appearing in the denominator of $\mathcal{F}_+(x)$, see Eq. (2.6), cancels the nontrivial potential term $-2\mathrm{dn}^2x$ in the Lamé Hamiltonian $H_{0,0}$ via the equality $\frac{d^2}{dx^2}(\log\Theta(x)) = \mathrm{dn}^2x - \frac{\mathbf{E}}{\mathbf{K}}$, that results in the nonperiodic potential (4.3), (4.4); see Fig. 4. By the Darboux construction, the system $H_{0,1}$ has the same spectrum as the one-gap Lamé system except that it possesses an additional discrete level of energy ε_1^-. This is the eigenvalue of the bound state described by the normalizable nodeless wave function

$$\Psi_{0,1}^{;1}(x;\beta_1^-,C_1) = \frac{1}{\mathcal{F}_+(x;\beta_1^-,C_1)} \tag{4.6}$$

shown in Fig. 5, which is a zero mode of the operator $A_{0,1}^\dagger$. The nonzero lower index in the Hamiltonian and potential reflects here the property that the system possesses one bound state in the lower forbidden band.

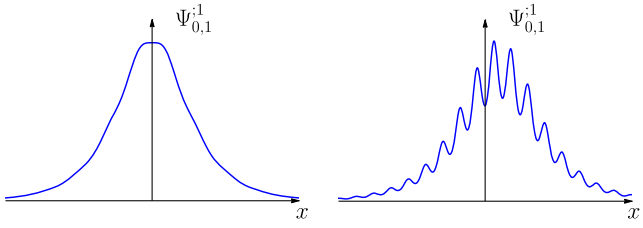

FIG. 5 (color online). The bound state eigenfunction of the system $H_{0,1}$. The state on the left corresponds to the potential $V_{0,1}$ with $C = 1$ in the central picture in Fig. 4. The state on the right, with $C > 1$, has energy closer to zero: when the energy modulus is lower, the state is broader, and the oscillations in it are well notable. By varying the parameter C, the soliton defect in the potential is displaced as well as the position of the bound state supported by it. In correspondence with this, in the case of $0 < C < 1$ not shown here, a localization of the wave function of the bound state is shifted to the $x < 0$ region in comparison with the case $C > 1$.

The upper index in notation for the wave function of the bound state is introduced bearing in mind a generalization for the case of a perturbed Lamé system with various bound states supported both in lower and upper forbidden bands.

Other physical and nonphysical eigenfunctions of $H_{0,1}$ are given by $A_{0,1}\Psi_\pm^\alpha(x)$. They correspond to the same permitted and prohibited values of energy as the eigenstates $\Psi_\pm^\alpha(x)$ of the periodic Lamé Hamiltonian. This shows that the introduced nonperiodic defect is reflectionless; physical Bloch states are transformed into the Bloch states.

Asymptotically, in the limit $x \to -\infty$, the potential has a form of the one-gap periodic Lamé potential, $V_{0,1}(x) \to V_{0,1}^{-\infty}(x) = V_{0,0}(x+\beta_1^-)$. In another limit $x \to +\infty$, we have $V_{0,1}(x) \to V_{0,1}^{+\infty}(x) = V_{0,0}(x-\beta_1^-)$. So, the defect produces a phase shift between the asymptotically periodic one-gap potentials that is equal to $-2\beta_1^-$. This observation follows also directly from (4.1). Asymptotically, we have $\mathcal{F}_+(x;\beta_1^-,C_1) \to C_1F(x;\beta_1^-)$ when $x \to -\infty$, and $\mathcal{F}_+(x;\beta_1^-,C_1) \to C_1^{-1}F(-x;\beta_1^-)$ for $x \to \infty$. Employing the results discussed below (3.5), we can write

$$A_{0,1}A_{0,1}^\dagger \to H_{0,0}(x \pm \beta_1^-) - \varepsilon_1^- \quad \text{for } x \to \mp\infty. \tag{4.7}$$

We get the phase displacement

$$\Delta\phi(\beta_1^-) = -2\beta_1^-, \qquad \varepsilon_1^- = -\mathrm{cd}^2\beta_1^- < 0, \tag{4.8}$$

where we indicate the discrete energy level of the bound state of $H_{0,1}$. The potential $V_{0,1}(x)$ may be treated as a soliton defect in the background of the one-gap periodic Lamé system.

Notice that in the limit $C_1 \to \infty$ (or $C_1 \to 0$) the soliton "goes" to infinity, and in correspondence with Eq. (4.3), $H_{0,1}$ transforms into the shifted Lamé Hamiltonian $H_{0,0}(x+\beta_1^-)$ [or $H_{0,0}(x-\beta_1^-)$].

Before we proceed further, let us show that the infinite-period limit of the obtained system with a periodicity defect corresponds to a reflectionless system of a generic form (2.17) with two bound states of energies $\mathcal{E}_0 = 0$ and $\mathcal{E}_1 = 1 - \kappa_1^2 < 0$. To this aim, we apply the limit $k \to 1$

to the operator (4.2). The nonphysical eigenfunction $\mathcal{F}_+(1)$ of the Lamé system in this limit transforms into the eigenfunction (2.16), whose explicit form is

$$\varphi_1(x;\kappa_1,\tau_1) = \frac{1}{\cosh x} W(\cosh x, \sinh \kappa_1(x+\tau_1)). \tag{4.9}$$

Indeed, in the indicated limit $Z(\beta|1) = \tanh\beta$, and $\mathrm{z}(\beta_1^-)$, defined in Table I, reduces to $\mathrm{z}(\beta_1^-) \to \tanh\beta_1^- + \frac{1}{\sinh\beta_1^- \cosh\beta_1^-} = \mathrm{cotanh}\,\beta_1^- \equiv \kappa_1$, where $1 < \kappa_1 < \infty$ since $\mathbf{K} \to \infty$, and then $\beta_1^- \in (0,\infty)$. We have also $\frac{\Theta(x\pm\beta|1)}{\Theta(x|1)} = \frac{\cosh(x\pm\beta)}{\cosh x}$. Introducing the notation $C_1 \equiv \exp\kappa_1\tau_1$, where τ_1 is an arbitrary real parameter, we find that $\mathcal{F}_+(1)$ transforms into $\frac{1}{\cosh x}(\cosh(x+\beta_1^-)\exp(-\kappa_1(x+\tau_1)) + \cosh(x-\beta_1^-)\exp(\kappa_1(x+\tau_1)))$. This function reduces, up to inessential nonzero multiplicative constant $\sinh\beta_1^-$, to (4.9). Then, in correspondence with the discussion of Sec. II B, the limit of the operator (4.2) is the Darboux generator, which intertwines the reflectionless Pöschl–Teller Hamiltonian (2.7) with the Hamiltonian operator (2.17). Thus, we conclude that the infinite-period limit of (4.3) corresponds to the reflectionless system (2.17).

To introduce several discrete energies into the spectrum of the one-gap Lamé system by making use of its nonphysical states from the lower prohibited band, consider first the case of the two bound states. It is not difficult to show that the Wronskian $W(\mathcal{F}_+(1),\mathcal{F}_-(2)) = \mathcal{F}_+(1)\mathcal{F}'_-(2) - \mathcal{F}'_+(1)\mathcal{F}_-(2)$, where $\mathcal{F}_+(1) = \mathcal{F}_+(x;\beta_1^-,C_1)$, $\mathcal{F}_-(2) = \mathcal{F}_-(x;\beta_2^-,C_2)$, takes strictly negative values, $W(x) < 0$, if $\mathbf{K} > \beta_1^- > \beta_2^- > 0$; see the Appendix. The corresponding energies of the nonphysical eigenstates of $H_{0,0}$ are ordered then as $0 > \mathcal{E}(\beta_1^- + i\mathbf{K}') > \mathcal{E}(\beta_2^- + i\mathbf{K}')$. With such a choice of the states, we can construct the Darboux–Crum transformation producing a nonperiodic deformation of Lamé system, which in addition to the one-gap spectrum of $H_{0,0}(x)$ has two discrete energy values $\varepsilon_j^- = \mathcal{E}(\beta_j^- + i\mathbf{K}')$, $j = 1,2$,

$$H_{0,2} = -\frac{d^2}{dx^2} + V_{0,2}(x),$$
$$V_{0,2}(x) = V_{0,0}(x) - 2\frac{d^2}{dx^2}(\log W(\mathcal{F}_+(1),\mathcal{F}_-(2))). \tag{4.10}$$

The discrete energy levels ε_1^- and ε_2^- correspond, respectively, to the two bound states

$$\Psi^{;1}_{0,2}(x;\beta_1^-,C_1,\beta_2^-,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\mathcal{F}_-(1))}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))}, \tag{4.11}$$

$$\Psi^{;2}_{0,2}(x;\beta_1^-,C_1,\beta_2,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\mathcal{F}_+(2))}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))}. \tag{4.12}$$

Other physical and nonphysical eigenstates of the system (4.10) are given by

$$\Psi^{\alpha}_{0,2;\pm}(x;\beta_1^-,C_1,\beta_2^-,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\Psi^{\alpha}_{\pm})}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))} \tag{4.13}$$

and correspond to the Darboux–Crum mapping of the eigenstates (2.2) of the initial Lamé system. The energies of these states are defined by the values of the parameter α exactly in the same way as for the system (2.1). In accordance with (4.1), expressions (4.11) and (4.12) for the bound states correspond to linear combinations of the eigenstates (4.13) with $\alpha = \beta_1^- + i\mathbf{K}'$ and $\alpha = \beta_2^- + i\mathbf{K}'$, respectively.

Let us take now n states

$$\mathcal{F}_{s_j}(j) = \mathcal{F}_{s_j}(x;\beta_j^-,C_j) \quad \text{with} \quad \mathbf{K} > \beta_1^- > \beta_2^- > \ldots > \beta_n^- > 0, \tag{4.14}$$

where s_j corresponds to a linear combination of the form (4.1) with index $+(-)$ for j odd (even). Then, by applying the Darboux–Crum construction on the basis of these eigenstates, we obtain a nonperiodic deformation $H_{0,n}$ of the Lamé system $H_{0,0}$ with n bound states with energies $0 > \varepsilon_1^- > \varepsilon_2^- > \ldots \varepsilon_n^- > -\infty$.

The potential of this system is given by a generalization of Eq. (4.10), in which the Wronskian has to be changed for

$$\mathbb{W}_{0,n}(x) = W(\mathcal{F}_+(1), \quad \mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n)). \tag{4.15}$$

The n bound states of energies ε_j^- are described by the normalizable wave functions

$$\Psi^{;j}_{0,n}(x;\beta_1^-,C_1,\ldots,\beta_n^-,C_n) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n),\mathcal{F}_{-s_j}(j))}{\mathbb{W}_{0,n}}, \quad j = 1,\ldots,n, \tag{4.16}$$

while other corresponding eigenstates of $H_{0,n}$ are given by the generalization of Eq. (4.13),

$$\Psi^{\alpha}_{0,n;\pm}(x;\beta_1^-,C_1,\ldots,\beta_n^-,C_n) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n),\Psi^{\alpha}_{\pm})}{\mathbb{W}_{0,n}}. \tag{4.17}$$

As in the case (4.10), bound states (4.16) may be obtained from (4.17) by putting there $\alpha = \beta_j^- + i\mathbf{K}'$, $j = 1,\ldots,n$,

and changing the wave functions $\Psi^{\alpha}_{\pm}$ on the rhs for the corresponding linear combinations of them.

Applying then the limit $x \to -\infty$ to the Wronskian $\mathbb{W}_{0,n}(x)$, we find that it transforms, up to a multiplicative constant, into $\mathbb{W}_{0,n}(x) = W(F(x;\beta_1^-)(x), \ldots, F(x;\beta_n^-))$. Asymptotically, we get a potential $V^{-\infty}_{0,n}(x) = \lim_{x\to-\infty}(-2\frac{d^2}{dx^2}\log \mathbb{W}_{0,n}(x)) = V_{0,0}(x+b)$, where $b = \sum_{j=1}^{n}\beta_j$. Analogously, in another limit $x \to +\infty$, we get the asymptotic form of the potential $V^{+\infty}_{0,n}(x) = V_{0,0}(x-b)$. The phase displacement produced by the n solitons (defects) is

$$\Delta\phi(\beta^-) = -2\sum_{j=1}^{n}\beta_j^-, \tag{4.18}$$

which generalizes the one-soliton effect (4.8).

The eigenstates of the system $H_{0,n}$ (4.16) and (4.17) can be presented in an alternative form [30],

$$\Psi(x;\beta_1^-, C_1, \ldots, \beta_n^-, C_n) = \mathbb{A}_{0,n}\Psi(x),$$
$$\mathbb{A}_{0,n} = A_{0,n}A_{0,n-1}\ldots A_{0,1}, \tag{4.19}$$

where the wave function on the lhs corresponds to (4.16) for the choice $\Psi = \mathcal{F}_{-s_j}(j)$ on the rhs, while it corresponds to the eigenfunctions (4.17) for the choice $\Psi = \Psi^{\alpha}_{\pm}$ on the rhs. The operator $\mathbb{A}_{0,n}$ is a differential operator of order n, which is constructed in terms of the recursively defined first-order differential operators (4.2) and

$$A_{0,j} = (\mathbb{A}_{0,j-1}\mathcal{F}_{s_j}(j))\frac{d}{dx}\frac{1}{(\mathbb{A}_{0,j-1}\mathcal{F}_{s_j}(j))}$$
$$= \frac{d}{dx} + \mathcal{W}_{0,j}, \qquad j = 2, \ldots, \tag{4.20}$$

where

$$\mathcal{W}_{0,j} = \Omega_{0,j} - \Omega_{0,j-1}, \qquad \Omega_{0,j} = -(\log \mathbb{W}_{0,j})_x, \tag{4.21}$$

and $\mathbb{W}_{0,1} \equiv \mathcal{F}_+(1)$. Equations (4.20) and (4.21) can also be used for $j = 1$ by putting $\mathbb{W}_{0,0} = 1$. Note here that, making use of Eqs. (4.19), it is easy to see that in the case of the two-soliton defect, particularly, the bound states (4.12) and (4.11) are reduced modulo multiplicative constants to the functions $\mathcal{F}_+(1)/\mathbb{W}_{0,2}$ and $\mathcal{F}_-(2)/\mathbb{W}_{0,2}$, respectively. This shows explicitly that the first function describing the discrete ground state is nodeless, while the second wave function corresponding to the first excited bound state has exactly one zero as it should be for the lowest bound states in the spectrum.

Relation (4.19) means that the operator $\mathbb{A}_{0,n}$ maps the eigenstates of the Lamé system (2.1) into the corresponding eigenstates of $H_{0,n}$. Its n-dimensional kernel is spanned by the eigenstates $\mathcal{F}_{s_j}(j)$, $j = 1, \ldots, n$. These relations reflect the fact that the Darboux–Crum transformation of order n corresponds to a composition of n subsequent Darboux maps $H_{0,0} \to H_{0,1} \to \ldots \to H_{0,n}$. In accordance with this, the operators $\mathbb{A}_{0,n}$ and $\mathbb{A}^{\dagger}_{0,n}$ intertwine the Hamiltonian operator $H_{0,n}(x)$ with the Lamé Hamiltonian $H_{0,0}(x)$,

$$\mathbb{A}_{0,n}H_{0,0} = H_{0,n}\mathbb{A}_{0,n}, \qquad \mathbb{A}^{\dagger}_{0,n}H_{0,n} = H_{0,0}\mathbb{A}^{\dagger}_{0,n}. \tag{4.22}$$

The products of the operator $\mathbb{A}_{0,n}$ and its conjugate are

$$\mathbb{A}_{0,n}\mathbb{A}^{\dagger}_{0,n} = \prod_{j=1}^{n}(H_{0,n} - \varepsilon_j^-), \qquad \mathbb{A}^{\dagger}_{0,n}\mathbb{A}_{0,n} = \prod_{j=1}^{n}(H_{0,0} - \varepsilon_j^-). \tag{4.23}$$

Alternative representation given by Eqs. (4.19) and (4.20) is valid for arbitrary Darboux–Crum transformations generated on the basis of n eigenstates of a generic Schrödinger Hamiltonian [30]. In the particular case of the one-gap Lamé system $H = H_{0,0}$ and the choice of eigenstates $\psi_j(x) = \Psi^{\alpha_j}_{+}(x)$, each of which, as we saw in the previous section, generates the translation of the Lamé system for $\alpha_j + i\mathbf{K}'$, we obtain the Darboux–Crum transformation producing the translation of $H_{0,0}(x)$ for $\sum_{j=1}^{n}\alpha_j + in\mathbf{K}'$. Taking into account that the system (2.1) besides the real period $2\mathbf{K}$ possesses also the imaginary period $2i\mathbf{K}'$, the shift produced by the Darboux–Crum transformation reduces to $\sum_{j=1}^{2r}\alpha_j$ in the case of even $n = 2r$ and to $\sum_{j=1}^{2r+1}\alpha_j + i\mathbf{K}'$ when $n = 2r+1$ is odd. Making use of this observation, it is obvious that when the total shift produced by the Darboux–Crum transformation reduces to a nontrivial period $2\mathbf{K}n_1 + 2i\mathbf{K}'n_2$ of the system (2.1) with $n_1^2 + n_2^2 \geq 2$, the corresponding higher-order generator $\mathbb{A}_n$ gives us the integral (multiplied in a generic case by a polynomial in $H_{0,0}$ [47]) of the one-gap Lamé system. This is the analog of the integral (2.13) of the reflectionless Pöschl–Teller system (2.7), which is the Lax–Novikov integral $\mathcal{P}_{0,0}$ for the system (2.1),

$$i\mathcal{P}_{0,0} = \frac{d^3}{dx^3} + (1 + k^2 - 3k^2\mathrm{sn}^2x)\frac{d}{dx} - 3k^2\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x. \tag{4.24}$$

In the limit $k \to 1$, it transforms into (2.13). The kernel of this third-order differential operator is spanned by eigenfunctions $\mathrm{dn}x$, $\mathrm{cn}x$, and $\mathrm{sn}x$, which correspond to the edges of the allowed bands. In correspondence with this, it admits an infinite number of factorizations. Particularly, it can be presented in the form

$$i\mathcal{P}_{0,0} = A_{1/\mathrm{cn}\,x}A_{\mathrm{cn}\,x/\mathrm{dn}\,x}A_{\mathrm{dn}\,x}, \tag{4.25}$$

where $A_{\mathrm{dn}\,x}$ is defined by relation of the form (3.1) with $\psi(x) = \mathrm{dn}\,x$, etc.

The sense of the factorization (4.25) is the following. The first factor on the right, $A_{\mathrm{dn}\,x}$, in accordance with its definition, annihilates dn$x$, the lower edge state of the valence band that is proportional to the limit of $F(x;\beta^-)$ for $\beta^- = \mathrm{K}$. Acting on the wave function sn$x$, which corresponds to the lower edge of the conduction band, the operator $A_{\mathrm{dn}\,x}$ translates it, as well as all other eigenstates of the Lamé system, for the half-period $\mathbf{K}$, $\mathrm{sn}(x+\mathbf{K}) = \mathrm{cn}\,x/\mathrm{dn}\,x$, and then this sn function with a shifted argument is annihilated by the operator $A_{\mathrm{cn}\,x/\mathrm{dn}\,x}$. Acting on the wave function cn$x$, which describes the upper edge state of the valence band, the $A_{\mathrm{dn}\,x}$ transforms it into $\mathrm{cn}(x+\mathbf{K})$, while the subsequent action of the $A_{\mathrm{cn}\,x/\mathrm{dn}\,x}$ transforms this into $\mathrm{cn}(x+\mathbf{K}+i\mathbf{K}') = -ik'/k\,\mathrm{cn}\,x$, which is annihilated finally by the first-order operator $A_{1/\mathrm{cn}\,x}$. In a similar way, one can construct five other factorizations of $\mathcal{P}_{0,0}$ having a simple interpretation in terms of the Darboux transformations (translations) generated by the edge states. Relation (4.27) corresponds here to the Darboux–Crum transformation that generates the total shift for the nontrivial period $2\mathbf{K}n_1 + 2i\mathbf{K}'n_2$ with $n_1 = n_2 = 1$ in correspondence with the discussion presented above.

The Lamé system's integral $\mathcal{P}_{0,0}$ satisfies the Burchnall–Chaundy relation

$$\mathcal{P}_{0,0}^2 = H_{0,0}(H_{0,0} - k'^2)(H_{0,0} - 1), \tag{4.26}$$

which lies in the basis of the hidden bosonized nonlinear supersymmetry of the one-gap Lamé system [18]. The zeros of the third-order polynomials in $H_{0,0}$ correspond to the energies of the edges of the allowed bands of (2.1). In the limit $k \to 1$, (4.26) transforms into relation (2.14), in which the double factor H_1^2 originates from the first two factors in (4.26) and roots in the shrinking of the valence band.

By analogy with the Lax–Novikov integral (2.13) for the reflectionless Pöschl–Teller system with one bound state, we can find the analogous integral for the $H_{0,n}$ system,

$$\mathcal{P}_{0,n} = \mathbb{A}_{0,n}\mathcal{P}_{0,0}\mathbb{A}_{0,n}^{\dagger}, \qquad [\mathcal{P}_{0,n}, H_{0,n}] = 0, \tag{4.27}$$

which is the differential operator of the order $2n+3$. In correspondence with (4.26) and (4.23), it satisfies the Burchnal–Chaundy relation

$$\mathcal{P}_{0,n}^2 = H_{0,n}(H_{0,n} - k'^2)(H_{0,n} - 1)\prod_{j=1}^{n}(H_{0,n} - \varepsilon_j^-)^2. \tag{4.28}$$

The systems $H_{0,0}$ and $H_{0,n}$ can be intertwined not only by the operators $\mathbb{A}_{0,n}$ and $\mathbb{A}_{0,n}^{\dagger}$ but also by the operators

$$\mathbb{B}_{0,n} = \mathbb{A}_{0,n}\mathcal{P}_{0,n} \quad \text{and} \quad \mathbb{B}_{0,n}^{\dagger}. \tag{4.29}$$

B. Intermediate forbidden band

Let us consider the intermediate prohibited band (gap) and the linear combinations of eigenstates (2.2) in it,

$$\Phi_+(1) \equiv \Phi_+(x;\beta_1^+, C_1) = C_1\Psi_+^{\beta_1^+}(x) + \frac{1}{C_1}\Psi_-^{\beta_1^+}(x), \tag{4.30}$$

$$\Phi_-(2) \equiv \Phi_-(x;\beta_2^+, C_2) = C_2\Psi_+^{\beta_2^+}(x) - \frac{1}{C_2}\Psi_-^{\beta_2^+}(x), \tag{4.31}$$

where $0 < \beta_l^+ < \mathbf{K}$ and C_l, $l = 1,2$ are arbitrary real constants restricted by the condition $C_l > 0$. Taking into account relation (2.4), the linear combinations used here differ effectively in sign in comparison to those employed in (4.1). This is related to the fact that the eigenvalue $\mathcal{E}(\beta^- + i\mathbf{K}')$ is an increasing function of the real parameter β^- in the lower prohibited band, while $d\mathcal{E}(\beta^+)/d\beta^+ < 0$ in the intermediate, upper forbidden band. Both these functions have an infinite number of zeros on the real line. The choice of any of these two functions as the function ψ in operator (3.1) produces by means of the first-order Darboux transformation a singular partner for the system $H_{0,0}(x)$.

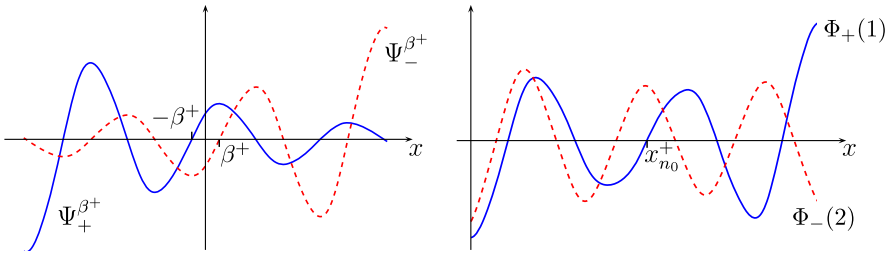

FIG. 6 (color online). Zeros of $\Psi_{\pm}^{\beta^+}(x)$ are in the equidistant points $2n\mathbf{K} \mp \beta^+$, and the amplitudes of these two functions increase exponentially in opposite directions. The amplitudes of the oscillating states $\Phi_{\pm}$ increase exponentially in both directions. The graphic on the right corresponds to the case $\beta_1^+ < \beta_2^+$.

Our next goal is to show how, by appropriate use of the second-order Darboux–Crum transformation applied to $H_{0,0}$, one can generate a regular system with two bound states in the gap.

Zeros of the nonphysical eigenfunctions $\Psi_+^{\beta^+}(x)$ are $-\beta^+ + 2n\mathbf{K}$, while the infinite set of zeros of the eigenstates $\Psi_-^{\beta^+}(x)$ is $\beta^+ + 2n\mathbf{K}$, $n \in \mathbb{Z}$. On the open intervals $(-\beta^+, \beta^+) + 2n\mathbf{K}$, functions $\Psi_+^{\beta^+}(x)$ and $\Psi_-^{\beta^+}(x)$ take nonzero values of the opposite sign, whereas on the open intervals $(\beta^+, 2\mathbf{K} - \beta^+) + 2n\mathbf{K}$, they take values of the same sign. Therefore, zeros of the linear combination (4.30) of $\Psi_+^{\beta_1^+}(x)$ and $\Psi_-^{\beta_1^+}(x)$ with $\beta_1^+ = \beta^+$ are inside the first of the indicated set of the open intervals, and zeros of (4.31) with $\beta_2^+ = \beta^+$ are inside the second set of the intervals. Since Φ_+ and Φ_- are linearly independent eigenstates of the same eigenvalue $\mathcal{E}(\beta^+)$, in correspondence with the oscillation theorem, each of the indicated open intervals contains exactly one zero of the respective function.

We want to generate a nontrivial nonsingular Darboux–Crum transformation based on the pair of the eigenfunctions (4.30) and (4.31). For this, the Wronskian of these functions should take nonzero nonconstant values. The choice

$$0 < \beta_1^+ < \beta_2^+ < \mathbf{K} \Leftrightarrow \mathcal{E}(\beta_1^+) > \mathcal{E}(\beta_2^+) \tag{4.32}$$

guarantees then that the intervals containing zeros of the functions (4.30) and (4.31) do not intersect, and between each two neighbor zeros x_n^+ and x_{n+1}^+ of the $\Phi_+(x;\beta_1^+, C_1)$, there will appear exactly one zero x_n^- of the $\Phi_-(x;\beta_2^+, C_2)$,

$$x_n^+ \in \mathcal{I}_n^+(1), \qquad x_n^- \in \mathcal{I}_n^-(2), \qquad \mathcal{I}_n^+(1) \cap \mathcal{I}_{n'}^-(2) = \emptyset, \tag{4.33}$$

where

$$\begin{aligned} \mathcal{I}_n^+(1) &= (-\beta_1^+, \beta_1^+) + 2n\mathbf{K}, \\ \mathcal{I}_n^-(2) &= (\beta_2^+, 2\mathbf{K} - \beta_2^+) + 2n\mathbf{K}. \end{aligned} \tag{4.34}$$

The amplitudes of the oscillating functions $\Psi_+^{\beta^+}(x)$ and $\Psi_-^{\beta^+}(x)$ increase exponentially for $x \to -\infty$ and $x \to +\infty$, respectively. As a consequence, in the limit $x \to +\infty$, the zeros x_n^+ tend to the right edges of the intervals $\mathcal{I}_n^+(1)$, while x_n^- tend to the left edges of the intervals $\mathcal{I}_n^-(2)$. In another limit $x \to -\infty$, the corresponding zeros tend to the opposite edges of the indicated intervals.

The Wronskian of the eigenfunctions (4.30) and (4.31) obeys the relation

$$\frac{d}{dx} W(y_1, y_2) = (\mathcal{E}(\beta_1^+) - \mathcal{E}(\beta_2^+)) y_1(x) y_2(x), \tag{4.35}$$

where $y_1 = \Phi_+(1)$ $y_2(x) = \Phi_-(2)$. From (4.35), it follows that zeros $x_n^\pm$ correspond exactly to the local extrema of the Wronskian. Let us choose a zero $x_{n_0}^+$ of y_1, $y_1(x_{n_0}^+) = 0$, such that $y_1'(x_{n_0}^+) > 0$. Then, in principle, we have two possibilities: either (i) $y_2(x_{n_0}^+) > 0$ or (ii) $y_2(x_{n_0}^+) < 0$. In case i, we find that $W(x_n^\pm) < 0$, while in case ii, we would have $W(x_n^\pm) > 0$ for any $n \in \mathbb{Z}$. Differentiation of (4.35) in x shows that in case i the zeros x_n^- and x_n^+ correspond to the local maxima and minima of the Wronskian, respectively. In case ii, the role of these zeros as local maxima and minima would be interchanged. Then, in case i, we conclude that the Wronskian takes strictly negative values for all x, while in case ii, it would be a strictly positive function. Though in both cases we would have a nodeless Wronskian, let us show that case i, illustrated on Fig. 6, is realized here. In the limits $x \to \pm\infty$, in correspondence with definition (4.30), (4.31), we have

$$\lim_{x\to+\infty} W(\Phi_+(1), \Phi_-(2)) = -\frac{1}{C_1 C_2} W(\Psi_-^{\beta_1^+}(x), \Psi_-^{\beta_2^+}(x)), \tag{4.36}$$

$$\lim_{x\to-\infty} W(\Phi_+(1), \Phi_-(2)) = C_1 C_2 W(\Psi_+^{\beta_1^+}(x), \Psi_+^{\beta_2^+}(x)). \tag{4.37}$$

Using these relations and the above-described behavior of the zeros of the functions $\Phi_+(1)$ and $\Phi_-(2)$ in the limit $x \to +\infty$, the corresponding local extrema values of W are given by

$$\begin{aligned} \lim_{x_n^\pm\to+\infty} W(x_n^\pm) = -\frac{1}{C_1 C_2} &\frac{\mathrm{H}'(0)\mathrm{H}(\beta_2^+ - \beta_1^+)}{\Theta^2(\beta_j)} \\ &\times \exp((\beta_j^+ + 2n\mathbf{K})(\mathrm{Z}(\beta_1^+) + \mathrm{Z}(\beta_2^+))), \\ n \gg 1, \end{aligned} \tag{4.38}$$

where $j = 1, 2$ and $\beta_1^+(\beta_2^+)$ corresponds here to $x_n^+(x_n^-)$. For the limits $x_n^\pm \to -\infty$, we have a similar expression with a unique change of the coefficient $1/(C_1 C_2)$ for $C_1 C_2$. Taking into account that $\mathrm{H}'(0) = \sqrt{2kk'\mathbf{K}/\pi} > 0$, and that $\mathrm{H}(\beta_2^+ - \beta_1^+) > 0$ because $0 < \beta_2^+ - \beta_1^+ < \mathbf{K}$, we conclude finally that $\mathbb{W}_{2,0}(x) = W(\Phi_+(1), \Phi_-(2))$ takes strictly negative values on all the real line. Additionally, we conclude that $-\mathbb{W}_{2,0}(x)$ blows up exponentially in both limits $x \to \pm\infty$.

Similarly to (4.10), we construct now the Hamiltonian

$$\begin{aligned} H_{2,0} &= -\frac{d^2}{dx^2} + V_{2,0}(x), \\ V_{2,0}(x) &= V_{0,0}(x) - 2\frac{d^2}{dx^2} \log W(\Phi_+(1), \Phi_-(2)). \end{aligned} \tag{4.39}$$

This quantum system has the same spectrum as the Lamé system except two additional discrete energy levels $\varepsilon_l^+ \equiv \mathcal{E}(\beta_l^+)$, $l = 1, 2$. These are described by the wave functions given by relations of the form (4.11), (4.12) with $\mathcal{F}_\pm(j)$ there changed for corresponding functions $\Phi_\pm(l)$. With some algebraic manipulations, the wave eigenfunctions can be presented in the form

$$\Psi_{2,0}^{1;}(x) = \text{const}\frac{\Phi_-(2)}{\mathbb{W}_{2,0}}, \qquad H_{2,0}\Psi_{2,0}^{1;}(x) = \varepsilon_1^+\Psi_{2,0}^{1;}(x), \tag{4.40}$$

$$\Psi_{2,0}^{2;}(x) = \text{const}\frac{\Phi_+(1)}{\mathbb{W}_{2,0}}, \qquad H_{2,0}\Psi_{2,0}^{2;}(x) = \varepsilon_2^+\Psi_{2,0}^{2;}(x). \tag{4.41}$$

The amplitude of these oscillating functions tends exponentially to zero in both limits $x \to \pm\infty$, which confirms their bound state nature; see Fig. 7. The relations (4.36) and (4.37) tell us that the Darboux–Crum transformation generated on the basis of the states appearing there on the right-hand sides produces a potential translated in $(\beta_1^+ + i\mathbf{K}') + (\beta_2^+ + i\mathbf{K}')$. Using this fact and taking into account the imaginary period $2i\mathbf{K}'$ of $V_{0,0}(x)$, we find that

$$V_{2,0}^{-\infty}(x) = \lim_{x\to-\infty} V_{2,0}(x) = V_{0,0}(x + \beta_1^+ + \beta_2^+),$$

and, analogously,

$$V_{2,0}^{+\infty}(x) = \lim_{x\to+\infty} V_{2,0}(x) = V_{0,0}(x - \beta_1^+ - \beta_2^+).$$

Therefore, similarly to the case of soliton defects corresponding to the bound states in the lower forbidden band, the two-soliton defect associated with the presence of the two bound states in the intermediate (upper) prohibited band produces the phase shift described by Eq. (4.18) with $n = 2$ and β_j^- there changed for β_l^+, where the parameters β_1^+ and β_2^+ obey the condition (4.32). The bound states here are described by infinitely oscillating wave functions, which have an infinite number of zeros and exponentially decreasing amplitudes. This situation contrasts with the bound states introduced into the lower forbidden band, where the wave functions are also exponentially decreasing but have a finite number of zeros, similarly to the nature of ordinary bound states.

The system (4.39) is also characterized by the Lax–Novikov integral, which in the present case is the differential operator of order 7,

$$\mathcal{P}_{2,0} = \mathbb{A}_{2,0}\mathcal{P}_{0,0}\mathbb{A}_{2,0}^\dagger, \qquad [\mathcal{P}_{2,0}, H_{2,0}] = 0. \tag{4.42}$$

The second-order operators $\mathbb{A}_{2,0}$ and $\mathbb{A}_{2,0}^\dagger$ intertwining the Lamé system $H_{0,0}$ with $H_{2,0}$ have the form (4.20) and (4.19) with the functions $\mathcal{F}_+(1)$ and $\mathcal{F}_-(2)$ changed here, respectively, for $\Phi_+(1)$ and $\Phi_-(2)$. They satisfy relations of the form (4.23) with $n = 2$, where $H_{0,n}$ has to be changed for $H_{2,0}$, and constants ε_j^- have to be changed for corresponding energy values ε_l^+, $l = 1, 2$, of the nonphysical eigenstates from the intermediate prohibited band we used in the construction.

Analogously to the discussion presented in the previous subsection, it is not difficult to show that the infinite-period limit applied to the system (4.39) corresponds to the reflectionless system given by potential (2.18).

The described procedure of the introduction of the periodicity defects with eigenvalues within the intermediate prohibited band can be generalized for the case of an arbitrary even number of the solitons. This can be done in a systematic way by choosing linear combinations of the wave functions of the form (4.30) and (4.31) with alternating lower indices $+$ and $-$, cf. (4.15), with the restriction on the parameters β^+, which generalizes that from (4.32),

$$0 < \beta_1^+ < \beta_2^+ < \ldots \beta_{2\ell}^+ < \mathbf{K} \Leftrightarrow \mathcal{E}(\beta_1^+) > \mathcal{E}(\beta_2^+) \ldots > \mathcal{E}(\beta_{2\ell}^+). \tag{4.43}$$

In the basis of such a construction, there is the property $|\mathbb{W}_{2\ell,0}(x)| > 0$ guaranteed by the choice (4.43), where $\mathbb{W}_{2\ell,0}(x)$ is the Wronskian of the corresponding 2ℓ nonphysical eingestates of the Lamé system,

$$\mathbb{W}_{2\ell,0}(x) = W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_+(2\ell - 1), \Phi_-(2\ell)). \tag{4.44}$$

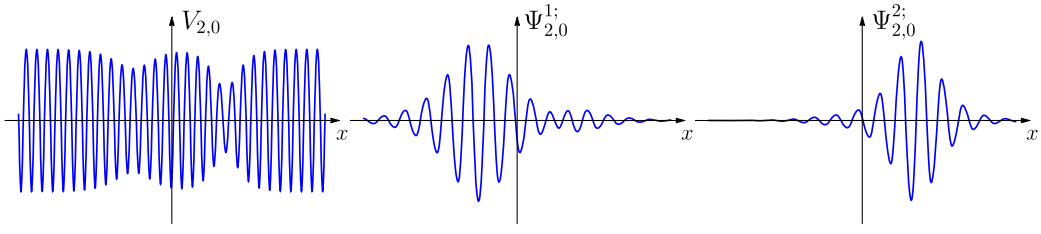

FIG. 7 (color online). Each of the two pulse-type bound states of the system $H_{2,0}$ is localized in one of the two periodicity defects of the potential $V_{2,0}$, which are showing up as compression modulations. The states also reveal a small tunnelling (asymmetry) in the direction of the other deformation.

The proof of this property is given in the Appendix.[5] As a generalization of (4.39) and (4.42), the Hamiltonian and Lax–Novikov integral are given here by the relations

$$H_{2\ell,0} = H_{0,0} - 2\frac{d^2}{dx^2}\log \mathbb{W}_{2\ell,0}, \tag{4.45}$$

$$\mathcal{P}_{2\ell,0} = \mathbb{A}_{2\ell,0}\mathcal{P}_{0,0}\mathbb{A}^{\dagger}_{2\ell,0}, \qquad [\mathcal{P}_{2\ell,0}, H_{2\ell,0}] = 0. \tag{4.46}$$

They satisfy the Burchnall–Chaundy relation of the form

$$\mathcal{P}^2_{2\ell,0} = H_{2\ell,0}(H_{2\ell,0} - k'^2)(H_{2\ell,0} - 1)\prod_{l=1}^{2\ell}(H_{2\ell,0} - \varepsilon_l^+)^2. \tag{4.47}$$

Here, $\varepsilon_l^+ = \mathcal{E}(\beta_l^+)$ are the eigenvalues of the bound states

$$\Psi^{l;}_{2\ell,0}(x;\beta_1^+, C_1, \ldots, \beta_{2\ell}^+, C_{2\ell}) = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \Phi_{(-1)^l}(l))}{\mathbb{W}_{2\ell,0}}, \quad l = 1, \ldots, 2\ell. \tag{4.48}$$

Other physical and nonphysical eigenstates of $H_{2\ell,0}$ of eigenvalues $\mathcal{E}(\alpha)$ are given by

$$\Psi^{\alpha}_{2\ell,0;\pm}(x;\beta_1^+, C_1, \ldots, \beta_{2\ell}^+, C_{2\ell}) = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \Psi^{\alpha}_{\pm})}{\mathbb{W}_{2\ell,0}}. \tag{4.49}$$

From this picture with even number $2\ell \geq 2$ of bound states in the intermediate forbidden band, one can obtain systems that contain odd number $2\ell - 1$ of discrete energy levels in the same prohibited band of the initial one-gap Lamé system. This can be achieved by sending any one of the 2ℓ solitons to infinity.

Let us see how this procedure works in the case of the system (4.39). For the sake of definiteness, we send the first soliton, associated with the higher discrete energy level $\mathcal{E}(\beta_1^+)$, to infinity. Another case corresponding to the limit associated with the soliton related to the lower discrete energy level can be realized in a similar way. To send the indicated soliton to infinity, we take a limit $C_1 \to \infty$. In analogous way, one can also consider the limit $C_1 \to 0$.

In the limit $C_1 \to \infty$, the potential $V_{2,0}(x)$ given by Eq. (4.39) transforms into

$$\lim_{C_1\to\infty} V_{2,0}(x) \equiv \breve{V}_{1,0}(x;\beta_1^+) = V_{0,0}(x) - 2\frac{d^2}{dx^2}\log W(\Psi_+^{\beta_1^+}, \Phi_-(2)). \tag{4.50}$$

The Hamiltonian $\breve{H}_{1,0}(x;\beta_1^+) = -\frac{d^2}{dx^2} + \breve{V}_{1,0}(x;\beta_1^+)$ possesses single bound state of energy ε_2^+, which can be obtained as a limit of the bound eigenstate $\Psi^{2;}_{2,0}(x)$ of $H_{2,0}$,

$$\lim_{C_1\to\infty} \Psi^{2;}_{2,0}(x) = \breve{\Psi}^{1;}_{1,0}(x); \tag{4.51}$$

see Fig. 8. In correspondence with the results of Sec. III, the Darboux transformation based on the single eigenfunction $\Psi_+^{\beta_1^+}(x)$ produces the Treibich–Verdier potential, $V_{0,0}(x) - 2\frac{d^2}{dx^2}\log \Psi_+^{\beta_1^+} = V_{0,0}(x + \beta_1^+ + i\mathbf{K}')$, and we can present (4.50) in the equivalent form

$$\breve{V}_{1,0}(x) = V_{0,0}(x + \beta_1^+ + i\mathbf{K}') - 2\frac{d^2}{dx^2}\left(\log\frac{W(\Psi_+^{\beta_1^+}, \Phi_-(2))}{\Psi_+^{\beta_1^+}}\right). \tag{4.52}$$

Function $W(\Psi_+^{\beta_1^+}, \Phi_-(2))/\Psi_+^{\beta_1^+}$ appearing in the argument of the logarithm is an eigenfunction of the system $H_{0,0}(x + \beta_1^+ + i\mathbf{K}')$. The Bloch-like eigenstates of this Hamiltonian operator can be obtained from the corresponding eigenstates of the Lamé system $H_{0,0}(x)$, $\Psi^{\alpha}_{\pm}(x + \beta_1^+ + i\mathbf{K}') = N_{\pm}(\alpha)\breve{\Psi}^{\alpha}_{\pm}(x + \beta_1^+)$, where

$$\breve{\Psi}^{\alpha}_{\pm}(x) = \frac{\Theta(x \pm \alpha)}{\mathrm{H}(x)}e^{\mp xZ(\alpha)} \tag{4.53}$$

and $N_{\pm}(\alpha) = \exp(\mp i(\frac{\alpha\pi}{2\mathbf{K}} + \mathbf{K}'Z(\alpha))$. Therefore, we have

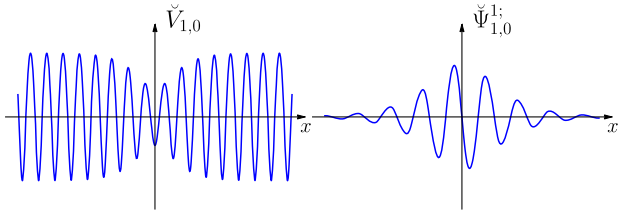

FIG. 8 (color online). Sending one soliton to infinity results in a potential supporting one bound state less. System $\breve{H}_{1,0}$ is related with the Lamé system $H_{0,0}$ by the Darboux–Crum transformation of the second order, while it is related with the singular Treibich–Verdier system by the first-order Darboux transformation. The symmetric state (presented by odd function here) is centered in the soliton deformation of the potential, and the tunnelling related to the soliton sent to infinity disappears.

[5]Like in the procedure shortly discussed in Sec. II B corresponding to the reflectionless Pöschl–Teller system, the defects also can be introduced in such a way that their associated energies will appear between the already placed discrete energy levels, but the final picture will be described equivalently by the Darboux–Crum transformation based on the Wronskian (4.44).

$$\frac{W(\Psi_+^{\beta_1^+}, \Psi_\pm^{\beta_2^+})}{\Psi_+^{\beta_1^+}} = C_\pm \breve{\Psi}_\pm^{\beta_2^+}(x+\beta_1^+). \tag{4.54}$$

Putting in both sides of the last relation $x=-\beta_1^+$ (or $x=\mp\beta_2^+$ to escape simple poles at both sides), we define the real nonzero constants $C_\pm$ in (4.54),

$$C_\pm = \mp \frac{\mathrm{H}(\beta_2^+ \mp \beta_1^+)\mathrm{H}'(0)}{\Theta(\beta_1^+)\Theta(\beta_2^+)} \exp\left(\pm\beta_1^+ \mathrm{Z}(\beta_2^+)\right). \tag{4.55}$$

Making a shift $x \to x-\beta_1^+$ in (4.50), all this gives us

$$V_{1,0}(x) \equiv \breve{V}_{1,0}(x-\beta_1^+) = 1 + k'^2 - 2\frac{\mathbf{E}}{\mathbf{K}} - 2\frac{d^2}{dx^2}\log \chi_{1,0}^{\beta_2^+}, \tag{4.56}$$

$$\chi_{1,0}^{\beta_2^+}(x) = \breve{C}_2 \Theta(x+\beta_2^+)\exp\left(-x\mathrm{Z}(\beta_2^+)\right) + \frac{1}{\breve{C}_2}\Theta(x-\beta_2^+)\exp\left(x\mathrm{Z}(\beta_2^+)\right). \tag{4.57}$$

Here, a real constant $\breve{C}_2$ is given in terms of C_2 by $\breve{C}_2 = C_2\sqrt{\frac{\mathrm{H}(\beta_2^+-\beta_1^+)}{\mathrm{H}(\beta_2^++\beta_1^+)}}\exp\left(\beta_1^+\mathrm{Z}(\beta_2^+)\right) > 0$, and we have taken into account the relation $\frac{d^2}{dx^2}\log \mathrm{H}(x) = \mathrm{dn}^2(x+i\mathbf{K}') - \frac{\mathbf{E}}{\mathbf{K}}$. In the limit $C_1 \to \infty$, the Wronskian in the denominator of the eigenstate (4.40) of energy $\mathcal{E}(\beta_1^+)$ of the system $H_{2,0}$ blows up exponentially, and this state disappears. On the other hand, the state (4.41) transforms into the bound state of energy $\mathcal{E}(\beta_2^+)$ of the system $H_{1,0}(x) = -\frac{d^2}{dx^2} + V_{1,0}(x)$,

$$\Psi_{2,0}^{2;}(x-\beta_1^+) \to \breve{\Psi}_{1,0}^{1;}(x-\beta_1^+) = \mathrm{const}\frac{\mathrm{H}(x)}{\chi_{1,0}^{\beta_2^+}(x)}. \tag{4.58}$$

The presence of this bound state in the spectrum of $H_{1,0}(x)$ is the unique difference in comparison with the spectrum of the one-gap Lamé system $H_{0,0}(x)$. The system $H_{1,0}(x)$ is related with $H_{0,0}(x)$, however, by the second-order Darboux–Crum transformation of the form (4.50) with x changed there for $x-\beta_1^+$. On the other hand, the system $H_{1,0}(x)$ can be related with the singular Treibich–Verdier system described by the potential $V_{0,0}(x+i\mathbf{K}')$, by the first-order Darboux transformation based on the function $\breve{\Psi}_\pm^\alpha(x-\beta_1^+)$ given by Eq. (4.53), which is the eigenfunction of the singular PT-invariant Hamiltonian operator $H_{0,0}(x+i\mathbf{K}')$. This picture is analogous to that for the Pöschl–Teller system when we want to introduce there the bound state between the already existing bound state and the continuous part of the spectrum; see Sec. II B.

In correspondence with the described picture, the system $H_{1,0}(x)$ is characterized by the irreducible Lax–Novikov integral

$$\mathcal{P}_{1,0}(x) = A_\psi \mathcal{P}_{0,0}(x+i\mathbf{K}')A_\psi^\dagger, \qquad \psi = \breve{\Psi}_+^{\beta_1^+}(x-\beta_1^+), \tag{4.59}$$

which is the differential operator of order 5, where $\mathcal{P}_{0,0}(x)$ is the Lax–Novikov integral (4.24) of the Lamé system $H_{0,0}(x)$. In (4.59), one can take, equivalently, $\psi = \Psi^{\beta_1^+}(x-\beta_1^+ + i\mathbf{K}')$.

Notice a remarkable similarity of the potential $V_{1,0}$ given by Eqs. (4.56) and (4.57) with the potential $V_{0,1}$ defined by Eqs. (4.4) and (4.5). The important difference of both potentials is, however, that $\mathrm{Z}(\beta_2^+)$ presents in the structure of $V_{1,0}$, while in the structure of the potential $V_{0,1}$, there appears $\mathrm{z}(\beta_1^-)$ defined in Table I. Unlike the nodeless bound state (4.6) of the system $V_{0,1}$, the bound state (4.58) of the system $V_{1,0}$ has an infinite number of zeros at $x_n = 2n\mathbf{K}$, and its amplitude, like that of the wave function (4.6), decreases exponentially as x goes to $\pm\infty$.

When $x \to \pm\infty$, Hamiltonian $H_{1,0}(x)$ asymptotically transforms into $H_{0,0}(x \mp \beta_2^+) - \mathcal{E}(\beta_2^+)$, and we get the phase displacement $\Delta\phi(\beta_2^+) = -2\beta_2^+$ generated by the one-soliton potential defect, which supports one bound state within the upper prohibited band of the original one-gap Lamé system.

Let us notice that one can also introduce an odd number of bound states into the gap by taking, instead of (4.32), the set of parameters $0=\beta_1^+ < \beta_2^+ < \ldots \beta_{2\ell}^+ < \mathbf{K}$, or $0 < \beta_1^+ < \beta_2^+ < \ldots \beta_{2\ell}^+ = \mathbf{K}$. This assumes the change of the state $\Phi_+(1)$ in Wronskian (4.44) for $\mathrm{sn}\,x$ in the first case, or $\Phi_-(2\ell)$ for $\mathrm{cn}\,x$ in the second case. Such alternatives, however, do not give anything new. They are reproduced just by taking, respectively, limits $\beta_1^+ \to 0$ or $\beta_{2\ell}^+ \to \mathbf{K}$ in the general picture presented in this subsection.

C. Bound states in both forbidden bands

One can introduce periodicity defects into the Lamé system by constructing the potentials that support bound states in both lower and upper forbidden bands. Similarly to the already discussed cases, the construction is based on the property that the Wronskian

$$\mathbb{W}_{2\ell,n}(x) = W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n)) \tag{4.60}$$

is a nodeless smooth function on all the real line; see the Appendix. In this way, the most general family of the one-gap Hamiltonians with $2\ell+n$ defects (solitons) introduced into the periodic background of Lamé potential $V_{0,0}(x)$ is defined by

$$H_{2\ell,n} = H_{0,0} - 2\frac{d^2}{dx^2}\log \mathbb{W}_{2\ell,n}(x). \tag{4.61}$$

The defects correspond to 2ℓ bound states in the spectral gap and n bound states in the lower prohibited band.

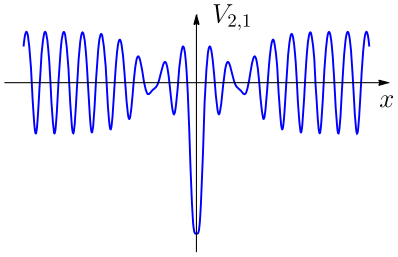

FIG. 9 (color online). Potential supporting two bound states in the gap and one bound state in the lower forbidden band. The defects in the form of the two compression modulations and a potential soliton well can be displaced arbitrarily in the periodic background.

In Fig. 9 is shown the form of the potential for the simplest case $\ell = n = 1$.

Each member of the family of Hamiltonians (4.61) possesses a nontrivial integral

$$\mathcal{P}_{2\ell,n} = \mathbb{A}_{2\ell,n}\mathcal{P}_{0,0}\mathbb{A}^{\dagger}_{2\ell,n}, \qquad [\mathcal{P}_{2\ell,n}, H_{2\ell,n}] = 0, \tag{4.62}$$

satisfying the relation

$$\mathcal{P}^2_{2\ell,n} = H_{2\ell,n}(H_{2\ell,n} - k'^2)(H_{2\ell,n} - 1) \times \prod_{l=1}^{2\ell}(H_{2l,n} - \varepsilon_l^+)^2 \prod_{j=1}^{n}(H_{2\ell,n} - \varepsilon_j^-)^2. \tag{4.63}$$

Here, $\mathbb{A}_{2\ell,n}$ is the differential operator of order $2\ell + n$, which is defined by $\mathbb{A}_{2\ell,n} = A_{2\ell,n}\ldots A_{2\ell,1}\mathbb{A}_{2\ell,0}$, where

$$A_{2\ell,j} = \frac{\mathbb{W}_{2\ell,j}}{\mathbb{W}_{2\ell,j-1}}\frac{d}{dx}\frac{\mathbb{W}_{2\ell,j-1}}{\mathbb{W}_{2\ell,j}}, \qquad j = 1,\ldots,n. \tag{4.64}$$

The first-order differential operator $A_{2\ell,n}$ and its conjugate generate the intertwining relations

$$A_{2\ell,n}H_{2\ell,n-1} = H_{2\ell,n}A_{2\ell,n}, \qquad A^{\dagger}_{2\ell,n}H_{2\ell,n} = H_{2\ell,n-1}A^{\dagger}_{2\ell,n} \tag{4.65}$$

and factorize the neighbor Hamiltonians $H_{2\ell,n}$ and $H_{2\ell,n-1}$ in the form

$$A_{2\ell,n}A^{\dagger}_{2\ell,n} = H_{2\ell,n} - \varepsilon_n^-, \qquad A^{\dagger}_{2\ell,n}A_{2\ell,n} = H_{2\ell,n-1} - \varepsilon_n^-. \tag{4.66}$$

The 2ℓ bound states of $H_{2\ell,n}$ of energies ε_l^+, $l = 1,\ldots,2\ell$, within the gap are given by

$$\Psi^{l;}_{2\ell,n} = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n), \Phi_{(-1)^l}(l))}{\mathbb{W}_{2\ell,n}}, \tag{4.67}$$

while the n bound states of energies ε_j^-, $j = 1,\ldots,n$, in the lower prohibited band have the form

$$\Psi^{;j}_{2\ell,n} = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n), \mathcal{F}_{-s_j}(j))}{\mathbb{W}_{2\ell,n}}. \tag{4.68}$$

Here, we do not indicate explicitly the parameters that define the functions $\Psi^{l;}_{2\ell,n}$ and $\Psi^{;j}_{2\ell,n}$ being in general of the form $\Psi(x;\beta_1^+, C_1^+, \ldots, \beta_{2\ell}^+, C_{2\ell}^+, \beta_1^-, C_1^-, \ldots \beta_n^-, C_n^-)$. Other, physical as well as nonphysical, eigenstates of $H_{2\ell,n}$ of eigenvalues $\mathcal{E}(\alpha)$ are given by

$$\Psi^{\alpha}_{2\ell,n;\pm} = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n), \Psi^{\alpha}_{\pm})}{\mathbb{W}_{2\ell,n}}. \tag{4.69}$$

It is always possible to eliminate any of the bound states from the spectrum taking the limit $C_r^{\pm} \to 0$, or $C_r^{\pm} \to \infty$ for the corresponding parameter. In the case we take such a limit for the parameter C_l^+ of the state $\Phi_{(-1)^{l+1}}(l)$, we obtain $H_{2\ell,n}(x) \to \breve{H}_{2\ell-1,n}(x;\beta_l^+)$, where $\breve{H}_{2\ell-1,n}(x;\beta_l^+)$ is the Hamiltonian of the system with $2\ell - 1$ bound states in the gap. Similarly to the case discussed in the previous subsection, the $\breve{H}_{2\ell-1,n}(x;\beta_l^+)$ can also be obtained by the Darboux–Crum transformation of order $2\ell - 1 + n$ applied to the singular Treibich–Verdier system. The Lax–Novikov integral $\breve{\mathcal{P}}_{2\ell-1,n}(x;\beta_l^+)$ of $\breve{H}_{2\ell-1,n}(x;\beta_l^+)$ appears from (4.62) via the indicated limit through the reduction, $\mathcal{P}_{2\ell,n}(x) \to (\breve{H}_{2\ell-1,n}(x;\beta_l^+) - \varepsilon_l^+)\breve{\mathcal{P}}_{2\ell-1,n}(x;\beta_l^+)$. On the other hand, if we take one of the two specified limits for the

parameter C_j^-, we obtain the Hamiltonian $\tilde{H}_{2\ell,n-1}(x;\beta_j^-)$, which corresponds to the system $H_{2\ell,n-1}(x)$ of the form (4.61) with the displaced argument, $x \to x+\beta_j^-$. The initial parameters β_i^- with $i = j+1,\ldots,n$ transform into the parameters β_i^-, $i = j,\ldots,n-1$, of the resulting system, and the same happens with the corresponding parameters C_i^-. Moreover, all parameters $C^\pm$ undergo rescaling, $C_l^+ \to c_l^+(\beta_l^+,\beta_j^-)C_l^+$, $l = 1,\ldots,2\ell$, $C_i^- \to c_i^-(\beta_i^-,\beta_j^-)C_i^-$, $i = 1,\ldots,n-1$, where $c_l^+ > 0$ and $c_i^- > 0$ are some functions of the indicated arguments, whose explicit form we do not write down in detail here.

Notice that in the most general case of one-gap quantum system $H_{2\ell-m,n} = -\frac{d^2}{dx^2} + V_{2\ell-m,n}(x)$ supporting $2\ell - m + n \geq 1$ bound states, relation (4.18) is generalized for

$$\Delta\phi = -2\sum_{j=1}^{n}\beta_j^- - 2\sum_{l=1}^{2\ell-m}\beta_l^+, \tag{4.70}$$

where $n \geq 0$, $2\ell - m \geq 0$, $m = 0,1$, and the omission of the corresponding sum is assumed when $n = 0$ or $\ell = 0$. This is the net phase displacement between $x = +\infty$ and $x = -\infty$ periodic asymptotics of the potential $V_{2\ell-m,n}(x)$, which is the one-gap Lamé potential $V_{0,0}(x)$ perturbed by $n \geq 0$ soliton defects of the potential well type and $2\ell - m \geq 0$, periodicity defects of the compression modulations nature.

In conclusion of this section, let us note that the notion of Hill's discriminant (Lyapunov function) is defined for a Schrödinger equation with periodic potential, and reflects coherently the properties of the eigenstates under the shift of the quantum system for its period [23,41]. The Darboux–Crum transformations that do not violate the periodicity of the potential produce isospectral systems and do not change the corresponding discriminants [42,43]. The systems we constructed here are *almost isospectral* to the one-gap Lamé system. Their potentials are not periodic functions, and so Hill's discriminant cannot be defined for them in a usual way. It can be considered only in the regions $x \to -\infty$ and $x \to +\infty$, where the periodicity (with a relative phase displacement defect) is restored asymptotically. At the same time, it is necessary to bare in mind that the Lyapunov function reflects the stability properties of the points in the spectrum: for periodic quantum systems, two linearly independent Bloch–Floquet states correspond to all the points inside the allowed bands, while the edge points are treated as nonstable because there one of the two solutions is unbounded [41]. Since the periodicity defects we constructed introduce into the spectrum of the Lamé system only the discrete energy values corresponding to nondegenerate bound states, one can say that they do not change the properties of stability of the spectrum of the initial system.

V. EXOTIC SUPERSYMMETRY

According to the analysis presented above, any pair of the Hamiltonians $H_{2\ell_1-m_1,n_1}$ and $H_{2\ell_2-m_2,n_2}$, where $m_{1,2} = 0,1$, can be related by means of the two pairs of intertwining operators. One pair of mutually conjugate operators intertwines the Hamiltonians directly. Another pair has higher differential order and does the same job via a virtual periodic one-gap system. The operators of the second pair involve in their structure the Lax–Novikov integral of the Lamé system $H_{0,0}$, or of its analog corresponding to the singular on the real line Treibich–Verdier one-gap system. Each of the subsystems in the pair $(H_{2\ell_1-m_1,n_1}, H_{2\ell_2-m_2,n_2})$ is also characterized by its proper Lax–Novikov integral. As a result, if we consider the extended system given by the matrix 2×2 Schrödinger operator composed from the pair of the indicated Hamiltonians, it will be described not just by the $\mathcal{N} = 2$ linear or nonlinear supersymmetry as it would be expected for the ordinary pair of Darboux(–Crum) related quantum mechanical systems. Instead, as in the case of nonperiodic reflectionless systems, it will be characterized by an exotic nonlinear $\mathcal{N} = 4$ supersymmetric structure that involves the two nontrivial bosonic generators composed from the Lax–Novikov integrals of the subsystems.

From the perspective of physical applications, the most interesting case corresponds to the pairs of the Schrödinger Hamiltonians, which can be related by the mutually conjugate first-order Darboux intertwiners alongside with the pair of higher-order intertwiners. It is this case that we consider in this section in detail.

We start from the general discussion of the picture corresponding to a basic case, from which other cases can be obtained via certain limiting procedures. Then, we illustrate this by considering the simplest examples, which reveal all the peculiarities of the exotic supersymmetric structure.

A. Exotic supersymmetry with the first-order supercharges: Generic picture

The first-order differential operators $A_{2\ell,n}$ and $A_{2\ell,n}^\dagger$ intertwine the Hamiltonians $H_{2\ell,n-1}$ and $H_{2\ell,n}$,

$$A_{2\ell,n}H_{2\ell,n-1} = H_{2\ell,n}A_{2\ell,n}, \qquad H_{2\ell,n-1}A_{2\ell,n}^\dagger = A_{2\ell,n}^\dagger H_{2\ell,n}, \tag{5.1}$$

and factorize them,

$$A_{2\ell,n}^\dagger A_{2\ell,n} = H_{2\ell,n-1} - \varepsilon_n^-, \qquad A_{2\ell,n}A_{2\ell,n}^\dagger = H_{2\ell,n} - \varepsilon_n^-, \tag{5.2}$$

where $\varepsilon_n^- = \mathcal{E}(\beta_n^- + i\mathbf{K}')$. These relations allow us to consider the extended system described by the Hamiltonian

$$\mathcal{H}_{2\ell,n} = \begin{pmatrix} H_{2\ell,n-1} & 0 \\ 0 & H_{2\ell,n} \end{pmatrix} \tag{5.3}$$

and by the pair of matrix operators

$$S^1_{2\ell,n} = \begin{pmatrix} 0 & A^\dagger_{2\ell,n} \\ A_{2\ell,n} & 0 \end{pmatrix}, \qquad S^2_{2\ell,n} = i\sigma_3 S^1_{2\ell,n}. \tag{5.4}$$

Taking the trivial integral $\Gamma = \sigma_3$ as a $\mathbb{Z}_2$-grading operator, we identify $\mathcal{H}_{2\ell,n}$ as the bosonic operator, $[\Gamma, \mathcal{H}_{2\ell,n}] = 0$, and $S^a_{2\ell,n}$, $a = 1, 2$, as the fermionic ones, $\{\Gamma, S^a_{2\ell,n}\} = 0$. They generate a superalgebra of $\mathcal{N} = 2$ supersymmetric quantum mechanics,

$$[\mathcal{H}_{2\ell,n}, S^a_{2\ell,n}] = 0, \qquad \{S^a_{2\ell,n}, S^b_{2\ell,n}\} = 2\delta^{ab}(\mathcal{H}_{2\ell,n} - \varepsilon^-_n). \tag{5.5}$$

By the redifinition of the Hamiltonian via an additive shift, $\mathcal{H}_{2\ell,n} - \varepsilon^-_n \to \mathcal{H}_{2\ell,n}$, one can transform (5.5) into the standard form of $\mathcal{N} = 2$ superalgebra describing the system with the zero energy of the nondegenerate ground state appearing in the spectrum of the "lower" subsystem of the extended matrix system. Since the subsystems $H_{2\ell,n-1}$ and $H_{2\ell,n}$ possess the nontrivial Lax–Novikov integrals being differential operators of orders $4\ell + 2n + 1$ and $4\ell + 2n + 3$, the extended system (5.3) possesses also two nontrivial bosonic integrals that we define in the form

$$P^1_{2\ell,n} = \begin{pmatrix} (H_{2\ell,n-1} - \varepsilon^-_n)\mathcal{P}_{2\ell,n-1} & 0 \\ 0 & \mathcal{P}_{2\ell,n} \end{pmatrix},$$
$$P^2_{2\ell,n} = \sigma_3 P^1_{2\ell,n}. \tag{5.6}$$

We introduced here the additional factor in the upper component whereby the upper and lower components of these integrals are operators of the same differential order. The commutation relations

$$[\mathcal{H}_{2\ell,n}, P^a_{2\ell,n}] = 0, \qquad [P^a_{2\ell,n}, P^b_{2\ell,n}] = 0,$$
$$[P^1_{2\ell,n}, S^a_{2\ell,n}] = 0 \tag{5.7}$$

extend the superalgebraic relations (5.5) and show that the integral $P^1_{2\ell,n}$ is the bosonic central charge. On the other hand, the nontrivial commutator $[P^2_{2\ell,n}, S^a_{2\ell,n}]$ generates the second pair of the fermionic supercharges $Q^a_{2\ell,n}$, which are the matrix differential operators of the order $2(2\ell + n + 1)$. As we shall see, the anticommutator of $Q^a_{2\ell,n}$ with $Q^b_{2\ell,n}$ produces a polynomial in matrix Hamiltonian $\mathcal{H}_{2\ell,n}$, while the anticommutator of $Q^a_{2\ell,n}$ with $S^b_{2\ell,n}$ generates the central charge $P^1_{2\ell,n}$. The second bosonic integral $P^2_{2\ell,n}$ generates finally a kind of a rotation between the supercharges $S^a_{2\ell,n}$ and $Q^a_{2\ell,n}$.

Taking in (5.3) the limit $C^+_l \to \infty$ or $C^+_l \to 0$ with l chosen from the set $1, \ldots, 2\ell$, we obtain another extended system:

$$\breve{\mathcal{H}}_{2\ell-1,n} = \begin{pmatrix} \breve{H}_{2\ell-1,n-1} & 0 \\ 0 & \breve{H}_{2\ell-1,n} \end{pmatrix}. \tag{5.8}$$

As we saw, the application of the limits $C^+_l \to \infty$ or $C^+_l \to 0$ to the corresponding Lax–Novikov integrals of the subsystems produces the reducible operators. The irreducible nonsingular Lax–Novikov integrals of $\breve{H}_{2\ell-1,n-1}$ and $\breve{H}_{2\ell-1,n}$ have orders $4\ell + 2n - 1$ and $4\ell + 2n + 1$ and include in their structure the Lax–Novikov integral of the singular Treibich–Verdier one-gap system. The bosonic integrals $\breve{P}^a_{2\ell-1,n}$ of the extended matrix system (5.8) are constructed from $\breve{\mathcal{P}}_{2\ell-1,n-1}$ and $\breve{\mathcal{P}}_{2\ell-1,n}$ like in (5.6). Again, $\breve{P}^1_{2\ell-1,n}$ will play the role of the central charge of the nonlinear superalgebra, while the commutator $[\breve{P}^2_{2\ell-1,n}, \breve{S}^a_{2\ell-1,n}]$ will generate the second pair of the supercharges $\breve{Q}^a_{2\ell-1,n}$. The exotic superalgebra of the system (5.8) will have as a result a form similar to that for the system (5.3).

Let us change index n for $n + 1$ in (5.3) and take one of the two limits

$$\lim_{C^-_{n+1} \to 0,\infty} H_{2\ell,n+1}(x) = \tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1}), \tag{5.9}$$

where the upper and lower signs on the rhs correspond, respectively, to the 0 and ∞ cases. In such a limit, we get the extended system described by the Hamiltonian,

$$\tilde{\mathcal{H}}_{2\ell,n} = \begin{pmatrix} H_{2\ell,n} & 0 \\ 0 & \tilde{H}_{2\ell,n} \end{pmatrix}, \tag{5.10}$$

where $\tilde{H}_{2\ell,n}$ corresponds to one of the indicated limits, $\tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1})$. Here, we have used the definition of the functions (4.1) and have taken into account that for the function (2.6) the identity $F(-x; \beta^-) = F(x; -\beta^-)$ is valid. The initial subsystems $H_{2\ell,n}$ and $H_{2\ell,n+1}$ in (5.3) with n changed for $n + 1$ are related by the first-order intertwining operators $A_{2\ell,n+1}$ and $A^\dagger_{2\ell,n+1}$. Then, the pair of $H_{2\ell,n}(x)$ and $\tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1})$ in (5.10) is related by the first-order intertwining operators

$$X_{2\ell,n}(x; \mp\beta^-_{n+1}) \equiv \lim_{C^-_{n+1} \to 0,\infty} A_{2\ell,n+1}$$
$$= \frac{\hat{\mathbb{W}}_{2\ell,n}(F(x; \mp\beta^-_{n+1}))}{\mathbb{W}_{2\ell,n}} \frac{d}{dx} \frac{\mathbb{W}_{2\ell,n}}{\hat{\mathbb{W}}_{2\ell,n}(F(x; \mp\beta^-_{n+1}))} \tag{5.11}$$

and $X^\dagger_{2\ell,n}(x; \mp\beta^-_{n+1})$, where $\hat{\mathbb{W}}_{2\ell,n}(f(x)) \equiv W(\Phi_+(1), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1) \ldots \mathcal{F}_{s_n}(n), f(x))$. The subsystems in

(5.10) are completely isospectral, and the exotic supersymmetry in this case has a structure similar to that of the system (5.3). However, unlike (5.3), the system (5.10) is characterized by the spontaneously broken exotic supersymmetry, and this fact, as we shall see, is properly reflected by the "fine structure" of the nonlinear superalgebra.

Another interesting case that could be mentioned corresponds to the limit

$$\lim_{\beta_{n+1}^- \to \beta_n^-} H_{2\ell,n+1} = \lim_{\beta_{n+1}^- \to \beta_n^-} \tilde{H}_{2\ell,n}(x; \mp \beta_{n+1}^-) = H_{2\ell,n-1}. \tag{5.12}$$

However, if we apply such a limit to the system (5.3) with index n changed for $n+1$, we obtain just a system of the form $\mathcal{H}_{2\ell,n}$ but with the permuted upper and lower corresponding Hamiltonians.

B. Unbroken exotic supersymmetry

Consider now the simplest case of the extended systems (5.3) with $\ell = 0$, $n = 1$. Besides the first-order operators $A_{0,1}$ and $A_{0,1}^\dagger$, the pair of Hamiltonians $H_{0,0}$ and $H_{0,1}$ is intertwined by the differential operators of order 4, $B_{0,1} = A_{0,1}\mathcal{P}_{0,0}(x)$ and $B_{0,1}^\dagger$. The systems $H_{0,0}$ and $H_{0,1}$ are also characterized by the Lax–Novikov integrals $\mathcal{P}_{0,0}(x)$ and $\mathcal{P}_{0,1}(x) = A_{0,1}\mathcal{P}_{0,0}(x)A_{0,1}^\dagger$. Besides the integrals of the form (5.4) and (5.6), the extended matrix system is characterized also by the pair of the supercharges

$$Q_{0,1}^1 = \begin{pmatrix} 0 & B_{0,1}^\dagger \\ B_{0,1} & 0 \end{pmatrix}, \qquad Q_{0,1}^2 = i\sigma_3 Q_{0,1}^1. \tag{5.13}$$

The fermionic integrals $S_{0,1}^a$ and $Q_{0,1}^a$ and the bosonic integrals $P_{0,1}^a$ together with the Hamiltonian $\mathcal{H}_{0,1}$ generate the nonlinear superalgebra

$$\{S^a, S^b\} = 2\delta^{ab}(\mathcal{H} - \varepsilon_1^-),$$
$$\{Q^a, Q^b\} = 2\delta^{ab}(\mathcal{H} - \varepsilon_1^-)C_3(\mathcal{H}), \tag{5.14}$$

$$\{S^a, Q^b\} = 2\delta^{ab}P^1, \tag{5.15}$$

$$[P^2, S^a] = -2i\epsilon^{ab}(\mathcal{H} - \varepsilon_1^-)Q^b,$$
$$[P^2, Q^a] = -2i\epsilon^{ab}(\mathcal{H} - \varepsilon_1^-)C_3(\mathcal{H})S^b, \tag{5.16}$$

$$[P^1, Q^a] = 0, \qquad [P^1, S^a] = 0, \tag{5.17}$$

where $C_3(\mathcal{H}) = \mathcal{H}(\mathcal{H} - k'^2)(\mathcal{H} - 1)$, ϵ^{ab} is the antisymmetric tensor, $\epsilon^{12} = 1$, and for the sake of simplicity, we omit the lower indices. The unique nondegenerate state with energy $\mathcal{E} = \varepsilon_1^-$ appearing in the spectrum of subsystem $H_{0,1}$ is annihilated by the shifted Hamiltonian $\mathcal{H} - \varepsilon_1^-$ and by all the integrals S^a, Q^a, and P^a. This means that the exotic supersymmetry of the extended Schrödinger system is unbroken. The doubly degenerate energy values corresponding to the edges of the allowed bands of the subsystems are the zeros of the third-order polynomial appearing in the superalgebra structure: $C_3(\mathcal{E}) = 0$ for $\mathcal{E} = 0, k'^2, 1$. This reflects the property that the corresponding edge states of the subsystems are detected by the fourth-order supercharges Q^a as well as by the bosonic integrals P^a; all these operators annihilate them. One can also show that the physical eigenstates $\Psi_\pm^\alpha$ and $A_{0,1}\Psi_\pm^\alpha$ of the upper and lower subsystems inside their valence and conduction bands possessing the quasimomentum of the opposite sign (they correspond to the different lower indices of the Bloch states) are distinguished by the bosonic integrals P^a.

The second relation $[P^1, S^a] = 0$ from (5.17) can be rewritten as a nonlinear differential equation for the superpotential $\mathcal{W}_{0,1}(x)$ shown in Fig. 10, see Eq. (4.21). This corresponds here to the first equation of the stationary mKdV hierarchy, which can be associated with the extended system with one nonperiodic soliton defect introduced into the one-gap Lamé system. At the same time, the equation $[\mathcal{H}, P^1] = 0$ can be presented in the form of the nonlinear differential equations of the third order for the potentials $V_\pm(x) \equiv \mathcal{W}_{0,1}^2 \pm \mathcal{W}_{0,1}' + \varepsilon_1^-$. These equations correspond to the first equation of the stationary KdV hierarchy, which can be associated with the one-gap Lamé system itself and with its deformation $V_-(x)$ produced by the one-soliton defect introduced into the periodic background of the one-gap Lamé system.

The generic case of the extended systems (5.3) and (5.8) is described by the exotic nonlinear superalgebras of the

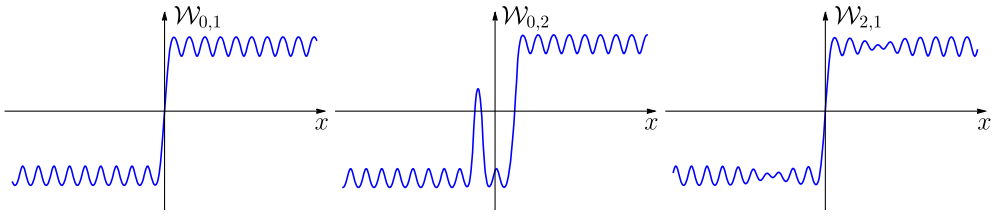

FIG. 10 (color online). Topological superpotentials in the form of the kink that incorporate one bound state into the spectrum. On the left is shown the superpotential that relates the systems $H_{0,0}$ with $H_{0,1}$. The superpotential in the center corresponds to the pair of the systems $H_{0,1}$ and $H_{0,2}$, while that on the right corresponds to the pair of $H_{2,0}$ and $H_{2,1}$.

same form. The unique difference is that the third-order polynomial $C_3(\mathcal{H})$ appearing here will be changed for the structure polynomials of the form (4.63), which are associated with the square of the corresponding Lax–Novikov integrals.

C. Spontaneously broken exotic supersymmetry

The case of the spontaneously broken exotic supersymmetry realized in the one-gap systems with the nonperiodicity defects can be illustrated by the extended system with the mutually displaced one-gap Lamé systems $H_{0,0}(x)$ and $\tilde{H}_{0,0}(x;\beta^-) = H_{0,0}(x+\beta^-)$. Though such systems are periodic, all the principle features of the structure of the exotic supersymmetry we observe in this case appear also in the extended systems composed from the completely isospectral systems with soliton defects.

The isospectral Hamiltonians $H_{0,0}(x)$ and $H_{0,0}(x+\beta^-)$ are connected by the first-order differential operator

$$X_{0,0}(x;\beta^-) = F(x;\beta^-)\frac{d}{dx}\frac{1}{F(x;\beta^-)} = \frac{d}{dx} + \Delta_{0,0}(x;\beta^-) \tag{5.18}$$

and by its Hermitian conjugate operator, where

$$\Delta_{0,0}(x,\beta^-) = \mathrm{Z}(x) - \mathrm{Z}(x+\beta^-) + \mathrm{z}(\beta^-) \tag{5.19}$$

is the superpotential shown in Fig. 11. To simplify notations, in what follows in this subsection, we omit lower indices in Hamiltonians, intertwining operators, and corresponding Lax–Novikov integrals and put $\beta^- = \beta$. Recall that $0 < \beta < \mathbf{K}$.

The operator (5.18) and its conjugate factorize the Hamiltonians,

$$\begin{aligned} X^\dagger(x;\beta)X(x;\beta) &= H(x) - \varepsilon(\beta), \\ X(x;\beta)X^\dagger(x;\beta) &= H(x+\beta) - \varepsilon(\beta), \end{aligned} \tag{5.20}$$

and intertwine them,

$$\begin{aligned} X(x;\beta)H(x) &= H(x+\beta)X(x;\beta), \\ X^\dagger(x;\beta)H(x+\beta) &= H(x)X^\dagger(x;\beta), \end{aligned} \tag{5.21}$$

where $\varepsilon(\beta) = \mathcal{E}(\beta + i\mathbf{K}') = -\mathrm{cs}^2\beta$. These first-order intertwining operators are related by $X^\dagger(x;\beta) = -X(x+\beta;-\beta)$, that follows from the identity $1/F(x;\beta) = F(x+\beta;-\beta)\exp(-\beta\mathrm{z}(\beta))$, and corresponds according to (5.11) to the limit $C^- \to \infty$ of the first-order operator $A_{0,1}$. In this limit, the topologically nontrivial superpotential $\mathcal{W}_{0,1}$ transforms into the topologically trivial superpotential $\Delta_{0,0}$, see Figs. 10 and 11. One can construct the second intertwiner being the differential operator of the order 2 by taking a composition of the two first-order intertwiners (5.18),

$$\mathcal{G}(x;\beta',\beta) = X(x+\beta';\beta-\beta')X(x;\beta'), \tag{5.22}$$

$\mathcal{G}(x;\beta',\beta)H(x) = H(x+\beta)\mathcal{G}(x;\beta,\beta')$, where we assume that $\beta' \neq \beta$. The first factor on the rhs in (5.22) intertwines the $H(x)$ with the Hamiltonian of the virtual system $H(x+\beta')$, and then this is intertwined by the second factor with $H(x+\beta)$. Notice also that $\mathcal{G}^\dagger(x;\beta',\beta) = \mathcal{G}(x+\beta;\beta'-\beta,-\beta)$.

One could think here that in this way intertwining operators of the higher order $n > 2$ can be constructed, but this is impossible because of the identity [22,47]

$$\mathcal{G}(x;\beta',\beta) = \mathcal{G}(x;\beta'',\beta) + G(\beta,\beta',\beta'')X(x;\beta), \tag{5.23}$$

from where it follows that the third-order differential operator

$$\begin{aligned} &X(x+\beta';\beta-\beta')\mathcal{G}(x;\beta'',\beta') \\ &\quad = -(H(x+\beta) - \varepsilon(\beta'-\beta))X(x;\beta) \\ &\qquad + G(\beta',\beta'',\beta)\mathcal{G}(x;\beta',\beta), \end{aligned} \tag{5.24}$$

which intertwines $H(x)$ and $H(x+\beta)$, reduces effectively to the first- and second-order intertwining operators $X(x;\beta)$ and $\mathcal{G}(x;\beta',\beta)$. Here, we used the notations $G(\beta,\beta',\beta'') \equiv g(\beta,-\beta') - g(\beta,-\beta'')$,

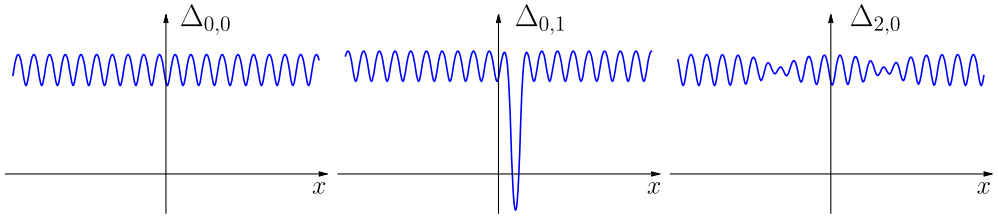

FIG. 11 (color online). Nontopological superpotentials that allow us to displace the periodic potential network of the Lamé system as well as the nonperiodic defects in it. The nontrivial displacements of the defects correspond to a nonlinear interaction between the soliton defects themselves and to their interaction with the periodic background (see Fig. 4). According to Fig. 10, these superpotentials are obtained by sending the kink, and the associated ground state of the Hamiltonian (5.3), to minus infinity, which generates the supersymmetry breaking. The shown superpotentials relate the following isospectral pairs: $H_{0,0}$ and $\tilde{H}_{0,0}$ (on the left), $H_{0,1}$ and $\tilde{H}_{0,1}$ (in the center), and $H_{2,0}$ and $\tilde{H}_{2,0}$ (on the right).

$$g(\beta,\beta') \equiv \mathrm{ns}\,\beta\,\mathrm{ns}\,\beta'\mathrm{ns}(\beta+\beta')(1-\mathrm{cn}\,\beta\,\mathrm{cn}\,\beta'\mathrm{cn}(\beta+\beta')). \tag{5.25}$$

The relation (5.23) reflects effectively a kind of the "gauge" nature of the parameter β', which appears in the structure of $\mathcal{G}(x;\beta',\beta)$ and is associated with a virtual system $H(x+\beta')$. On the other hand, from the same relation and definition (5.22), one finds that the second-order operator

$$Y(x;\beta) = X(x+\beta';\beta-\beta')X(x;\beta') - g(\beta,-\beta')X(x;\beta) \tag{5.26}$$

is invariant under the change $\beta' \to \beta''$. Thus, being a certain linear combination of (5.18) and (5.22), $Y(x;\beta)$ is the "gauge-invariant" second-order intertwining operator, $Y(x;\beta)H(x) = H(x+\beta)Y(x;\beta)$, which does not depend on the value of the virtual parameter in spite of its appearance on the rhs in (5.26). The conjugate operator acts in the opposite direction, and similarly to the first order intertwining operator, we have $Y^\dagger(x;\beta) = Y(x+\beta;-\beta)$.

One can represent $Y(x;\beta)$ in the explicitly β'-independent form in terms of the superpotential (5.19) and parameter β. However, we do not need here such an expression and will use the representation (5.26).

From the properties of $X(x;\beta)$ and $Y(x;\beta)$, it follows that the third-order operators $X^\dagger(x;\beta)Y(x;\beta)$ and $Y^\dagger(x;\beta)X(x;\beta)$ reduce, up to the additive constants, to the third-order Lax–Novikov integral $\mathcal{P}(x) = \mathcal{P}_{0,0}(x)$ given by Eq. (4.24) and to $\mathcal{P}(x+\beta)$, respectively. Namely, we have

$$\begin{aligned} X^\dagger(x;\beta)Y(x;\beta) &= -i\mathcal{P}(x) - \mathcal{N}_0(\beta), \\ X(x;\beta)Y^\dagger(x;\beta) &= i\mathcal{P}(x+\beta) - \mathcal{N}_0(\beta) \end{aligned} \tag{5.27}$$

and the pair of identity relations, which can be obtained from (5.27) by the Hermitian conjugation. The β-dependent constant $\mathcal{N}_0(\beta)$ is given by[6]

$$\mathcal{N}_0(\beta) = \mathrm{dn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{ns}^3\beta = \frac{1}{2}\frac{d}{d\beta}\varepsilon(\beta). \tag{5.28}$$

Similarly to (5.21), the second-order intertwining operators generate the second-order polynomial in the isospectral Hamiltonians,

$$\begin{aligned} Y^\dagger(x;\beta)Y(x;\beta) &= \mathcal{N}_2(H(x),\beta), \\ Y(x;\beta)Y^\dagger(x;\beta) &= \mathcal{N}_2(H(x+\beta),\beta), \end{aligned} \tag{5.29}$$

where

$$\mathcal{N}_2(H(x),\beta) = H^2(x) + c_1(\beta)H(x) + c_2(\beta), \tag{5.30}$$

$$\begin{aligned} c_1(\beta) &= -k'^2 - \mathrm{ns}^2\beta = \varepsilon(\beta) - 1 - k'^2, \\ c_2(\beta) &= \mathrm{dn}^2\beta\mathrm{ns}^4\beta = (\varepsilon(\beta)-1)(\varepsilon(\beta)-k'^2). \end{aligned} \tag{5.31}$$

Finally, for the products of the intertwining operators with the Lax–Novikov integral, we obtain

$$-iX(x;\beta)\mathcal{P}(x) = \mathcal{N}_1(H(x+\beta),\beta)Y(x;\beta) + \mathcal{N}_0(\beta)X(x;\beta) \tag{5.32}$$

$$i\mathcal{P}(x)X^\dagger(x;\beta) = \mathcal{N}_1(H(x),\beta)Y^\dagger(x;\beta) + \mathcal{N}_0(\beta)X^\dagger(x;\beta), \tag{5.33}$$

$$iY(x;\beta)\mathcal{P}(x) = \mathcal{N}_2(H(x+\beta),\beta)X(x;\beta) + \mathcal{N}_0(\beta)Y(x;\beta), \tag{5.34}$$

$$-i\mathcal{P}(x)Y^\dagger(x;\beta) = \mathcal{N}_2(H(x),\beta)X^\dagger(x;\beta) + \mathcal{N}_0(\beta)Y^\dagger(x;\beta) \tag{5.35}$$

and four other relations given by the Hermitian conjugation. Here, we introduce the notation

$$\mathcal{N}_1(H(x),\beta) = H(x) - \varepsilon(\beta). \tag{5.36}$$

The operators $X(x;\beta)$ and $Y(x;\beta)$ and their conjugate ones intertwine the Lax–Novikov integrals $\mathcal{P}(x)$ and $\mathcal{P}(x+\beta)$ exactly in the same way as they do this with the corresponding Hamiltonians.

Now, we are in a position to identify the superalgebra of the extended Schrödinger system $\tilde{\mathcal{H}} = \mathrm{diag}(H(x), H(x+\beta))$, which corresponds to (5.10) with $\ell = n = 0$ and lower component $\tilde{H}_{0,0}(x;\beta_1^-)$. This extended system is characterized by the two pairs of the fermion integrals $\tilde{S}^a(x;\beta)$ and $\tilde{Q}^a(x;\beta',\beta)$, constructed from the first-, $X^\dagger(x;\beta)$, $X(x;\beta)$, and second-order, $Y^\dagger(x;\beta)$, $Y(x;\beta)$, intertwining operators in the form similar to that in (5.4), and by the two boson integrals $\tilde{P}^1 = \mathrm{diag}(\mathcal{P}(x), \mathcal{P}(x+\beta))$ and $\tilde{P}^2 = \sigma_3\tilde{P}^1$. These 2×2 matrix operators generate the exotic nonlinear $\mathcal{N}=4$ superalgebra,

$$\begin{aligned} \{\tilde{S}^a,\tilde{S}^b\} &= 2\delta^{ab}\mathcal{N}_1(\tilde{\mathcal{H}},\beta), \\ \{\tilde{Q}^a,\tilde{Q}^b\} &= 2\delta^{ab}\mathcal{N}_2(\tilde{\mathcal{H}},\beta), \end{aligned} \tag{5.37}$$

$$\{\tilde{S}^a,\tilde{Q}^b\} = -2\epsilon^{ab}\tilde{P}^1 - 2\delta^{ab}\mathcal{N}_0(\beta), \tag{5.38}$$

[6]Notice here that, for the limit case $\beta = \mathbf{K}$, $\mathcal{N}_0(\mathbf{K}) = 0$. Then, for the choice $\beta' = \mathbf{K} + i\mathbf{K}'$, the coefficient g in (5.26) turns into zero, and the Hermitian conjugate form of the first relation in (5.27) corresponds to factorization (4.27). Another choice, for instance, $\beta' = i\mathbf{K}'$, gives a factorization $i\mathcal{P}_{0,0}(x) = A_{1/\mathrm{sn}\,x}A_{\mathrm{sn}\,x/\mathrm{dn}\,x}A_{\mathrm{dn}\,x}$.

$$[\tilde{P}^2, \tilde{S}^a] = -2i\mathcal{N}_1(\tilde{\mathcal{H}}, \beta)\tilde{Q}^a - 2i\mathcal{N}_0(\beta)\tilde{S}^a,$$
$$[\tilde{P}^2, \tilde{Q}^a] = 2i\mathcal{N}_2(\tilde{\mathcal{H}}, \beta)\tilde{S}^a + 2i\mathcal{N}_0(\beta)\tilde{Q}^a, \quad (5.39)$$

$$[\tilde{P}^1, \tilde{Q}^a] = 0, \qquad [\tilde{P}^1, \tilde{S}^a] = 0, \quad (5.40)$$

where $\mathcal{N}_1(\tilde{\mathcal{H}}, \beta)$ and $\mathcal{N}_2(\tilde{\mathcal{H}}, \beta)$ are defined as above with the operator argument $H(x)$ changed for $\tilde{\mathcal{H}}$. The matrix Hamiltonian operator $\tilde{\mathcal{H}}$ plays here, as well as in the superalgebra we considered in the previous subsection, the role of the central element. Note that the constants appearing in the structure of $\mathcal{N}_1(\tilde{\mathcal{H}}, \beta)$ and $\mathcal{N}_2(\tilde{\mathcal{H}}, \beta)$ correspond to the energies of the doubly degenerate states of the system at the edges of the allowed bands: $\mathcal{E} = 0, k'^2, 1$.

The sub-superalgebra generated by the supercharges $\tilde{S}^a$ and by the Hamiltonian $\tilde{\mathcal{H}}$ with $0 < \beta < \mathbf{K}$ corresponds to the case of the spontaneously broken linear (Lie) $\mathcal{N} = 2$ supersymmetry. The first-order supercharges do not annihilate the two ground states $\Psi^t_+ \equiv (\mathrm{dn}x, 0)$ and $\Psi^t_- \equiv (0, \mathrm{dn}(x+\beta))$ being eigenstates of zero energy of the extended system. This is obvious from the first relation from (5.37) and Eq. (5.36). The quantity $-\varepsilon(\beta) = \mathrm{cs}^2\beta > 0$ defines here the scale of supersymmetry breaking. The second relation from (5.37) and Eqs. (5.30) and (5.31) show that the second-order supercharges $\tilde{Q}^a$ also do not annihilate these states. These edge states, however, as well as the edge states of energies k'^2 and 1, which correspond to the two other doubly degenerate energy levels of $\tilde{\mathcal{H}}$, are zero modes of the bosonic generators $\tilde{P}^a$.

The limit case $\beta = \mathbf{K}$ corresponding to $\varepsilon = 0$ is special here. At $\beta = \mathbf{K}$, the coefficient $\mathcal{N}_0$ turns into zero, and the indicated two ground states are zero modes of the first-order supercharges. The structure of the nonlinear superalgebra (5.37)–(5.40) essentially simplifies because of the disappearance of the three terms in Eqs. (5.38) and (5.39). In this case, the second-order supercharges $\tilde{Q}^a$ annihilate the doubly degenerate states at the edges of the valence and conduction bands of energies k'^2 and 1. Since the second-order supercharges $\tilde{Q}^a$ do not annihilate the degenerate pair of the ground states in this case either, the extended system $\tilde{\mathcal{H}}$ with $\beta = \mathbf{K}$ is characterized by the partially broken exotic nonlinear $\mathcal{N} = 4$ supersymmetry.

Notice that, though at $\beta = \mathbf{K}$ the sub-supersymmetry $\mathcal{N} = 2$ generated by $\tilde{\mathcal{H}}$ and $\tilde{S}^a$ is unbroken, the subsystems $H(x)$ and $H(x+\beta)$ are completely isospectral, and the superextended system is characterized by the zero Witten index [48]. This is a characteristic peculiarity of the quantum supersymmetric systems composed from the periodic completely isospectral pairs, which was noted for the first time by Braden and Macfarlane [3] for the particular case of the pair of one-gap periodic Lamé systems shifted mutually for the half-period $\beta = \mathbf{K}$ and later was discussed in a more broad context of "self-isospectrality" by Dunne and Feinberg [11]. In the framework of the nonlinear "tri-supersymmetric" structure, it was analyzed then in Refs. [19,42].

In the context of the breaking of the exotic supersymmetry, it is worth noticing that, generally speaking, the second-order supercharges are not defined uniquely here. Instead of $\tilde{Q}^a$, one can take linear combinations of $\tilde{Q}^a$ and $\tilde{S}^a$, for instance, $\hat{Q}^a = \tilde{Q}^a + \gamma\tilde{S}^a$, where γ is a real constant. The particular choice $\gamma = \mathrm{dn}\beta/\mathrm{sn}\beta\,\mathrm{cn}\beta$ gives then the supercharges $\hat{Q}^a$, which satisfy the anticommutation relations $\{\hat{Q}^a, \hat{Q}^b\} = 2\delta^{ab}\tilde{\mathcal{H}}(\tilde{\mathcal{H}} + \varrho(\beta))$, where $\varrho(\beta) = k'^2\mathrm{sc}^2\beta$. Hence, for $\beta \neq \mathbf{K}$, the supercharges $\hat{Q}^a$ annihilate the ground states of zero energy of the system $\tilde{\mathcal{H}}$ (while other states from their kernels correspond to nonphysical eigenstates of $\tilde{\mathcal{H}}$). In this case, the exotic supersymmetry generated by $\tilde{S}^a$, $\hat{Q}^a$, $\tilde{P}^a$, and $\tilde{\mathcal{H}}$ should be interpreted as partially broken. However, the second-order supercharges $\hat{Q}^a$, unlike $\tilde{Q}^a$, are not defined for the limit case $\beta = \mathbf{K}$. The supercharges $\hat{Q}^a$ with the indicated choice of the parameter γ correspond to the second-order intertwining generators (5.22) with $\beta' = \mathbf{K}$.

As in the case of the unbroken exotic supersymmetry we considered in the previous subsection, the Lax–Novikov matrix integral $\tilde{P}^1$ plays here the role of the bosonic central charge, and the second relation in (5.40) corresponds to the stationary equation of the mKdV hierarchy for the topologically trivial superpotential $\Delta_{0,0}(x, \beta)$. The relation $[\tilde{\mathcal{H}}, \tilde{P}^1] = 0$ corresponds to the pair of stationary equations of the KdV hierarchy for the functions $V_\pm(x) = \Delta_{0,0}(x, \beta)^2 \pm \Delta'_{0,0}(x, \beta) + \varepsilon(\beta)$, which represent the potentials of the corresponding mutually shifted Schrödinger systems.

The superalgebra (5.37)–(5.40) in comparison with that of the unbroken exotic supersymmetry case (5.14)–(5.17) contains the terms with the coefficient $\mathcal{N}_0(\beta)$ in (5.38) and (5.39), which are absent in (5.15) and (5.16). There are also other obvious differences in these two forms of superalgebras, which reflect properly the unbroken and spontaneously broken character of the exotic supersymmetries and different topological nature of the corresponding superpotentials. At the formal level, some of these differences are associated with a nontrivial limit procedure applied to the fourth-order intertwining operators $B_{0,1} = A_{0,1}\mathcal{P}_{0,1}(x)$ and $B^\dagger_{0,1}$, in terms of which the fourth-order supercharges Q^a were constructed in the previous subsection. In correspondence with the limit (5.11), we have $B_{0,1} \to X(x;\beta)\mathcal{P}(x)$, $\mathcal{P}(x) = \mathcal{P}_{0,0}(x)$. But according to the relation (5.32), the fourth-order intertwining operator we obtain in the limit is reducible, and, finally, instead of the fourth-order intertwining operators, here we have the second-order operators $Y(x;\beta)$ and $Y^\dagger(x;\beta)$, which intertwine the completely isospectral pair of the Schrödinger systems $H(x) = H_{0,0}(x)$ and $H(x+\beta) = H_{0,0}(x+\beta)$.

VI. DISCUSSION AND OUTLOOK

To conclude, we summarize shortly the results and point out further possible research directions.

We showed how, by applying the Darboux–Crum transformations to the quantum one-gap Lamé system, an arbitrary countable number of bound states can be introduced into the forbidden bands of its spectrum. These states are trapped by localized perturbations of the periodic potential background of the initial system. The nature of the perturbations depends on whether they support discrete energy levels in the lower forbidden band, or in the finite gap separating the allowed valence and conduction bands. In the first case, the perturbations have a nature of the smooth soliton potential *wells* superimposed on the background of the Lamé system, while the discrete energy levels in the gap are supported by compression *modulations* of the periodic background. Though both types of perturbations have a soliton nature, to distinguish, we identify them here as the W-type and M-type defects, respectively. The nature of the bound states is essentially different in these two cases. The $n \geq 1$ bound states trapped by the W-type defects are described by the wave functions with finite number $0 \leq j \leq n-1$ of nodes on the real line. In contrast, the bound states supported by the M-type defects have an infinite number of nodes and represent oscillating trapped pulses.

The obtained nonperiodic systems are reflectionless; their physical states inside the valence and conduction bands are described by the Darboux–Crum transformed Bloch states of the Lamé system, just like the scattering states of quantum systems with multisoliton potentials are given by a Darboux–Crum transformation of free particle plane waves. Similarly to the multisoliton reflectionless potentials, which exponentially tend to a constant value corresponding to the free particle case, here the asymptotics of the perturbed potentials corresponds to the periodic one-gap Lamé potential. We show that the net phase displacement (defect) between $x = +\infty$ and $x = -\infty$ periodic asymptotics of the potential are given by a simple sum of the same parameters that determine, via the elliptic dn^2 parametrization, the discrete energy levels.

The procedure for introducing the W- and the M-type periodicity defects has some important differences. In the first case, the order n of the Darboux–Crum transformation corresponds exactly to the number of the introduced bound states. In the second case, the same is true when the number of discrete energy values is even. The odd number of the discrete energy levels in the gap is obtained by sending one of the already introduced 2ℓ M-type defects to infinity. The resulting potential with $2\ell - 1$ M defects is related to the initial Lamé system by 2ℓ-th-order Darboux–Crum transformation. At the same time, it can be related by the Darboux–Crum transformation of order $2\ell - 1$ with a singular one-gap Treibich–Verdier system obtained by a displacement of the regular Lamé system for one of its two complex half-periods. The indicated complex displacement can itself be generated by the first-order Darboux transformation. This explains the existence of two alternative Darboux–Crum transformations whose orders differ by 1.

The procedure described in this article allows us to construct the irreducible Lax–Novikov integrals of motion for the perturbed systems $H_{2\ell-m,n}$ via the Darboux–Crum dressing of the Lax–Novikov integral of the initial periodic Lamé system $H_{0,0}$. This is similar, again, to the situation with the transparent quantum systems described by multisoliton potentials, for which the Lax–Novikov integrals are the Darboux–Crum dressed form of the momentum operator of the free particle. The Lax–Novikov integrals here are differential operators of order $2(n + 2\ell - m) + 3$ for the system with $n \geq 0$ W-type and $2\ell - m \geq 0$, $m = 0, 1$, M-type defects. The condition of conservation of these integrals generates a nonlinear differential equation of order $2(n + 2\ell - m) + 3$ for the potential $V_{2\ell-m,n}(x)$. This ordinary nonlinear differential equation of odd order in the highest derivative belongs to the stationary KdV hierarchy.

For an extended system composed from an arbitrary pair of the Hamiltonians $H_{2\ell_1-m_1,n_1}$ and $H_{2\ell_2-m_2,n_2}$, which possess $n_i \geq 0$, $i = 1, 2$, discrete energy levels in the lower forbidden band, and $2\ell_i - m_i \geq 0$, $m_i = 0, 1$, bound states in the gap, the presence of the Lax–Novikov integrals has an essential consequence. The whole system is now described not just by an $\mathcal{N} = 2$ linear or nonlinear supersymmetry as would be expected in the case of a Darboux–Crum related pair of ordinary, nontransparent, or not periodic finite-gap, quantum Hamiltonians. Instead, such a system is characterized by an exotic nonlinear $\mathcal{N} = 4$ supersymmetry that, besides two pairs of the fermion supercharges of odd and even differential orders, involves two bosonic generators composed from the Lax–Novikov integrals of the subsystems. We investigated in more detail the most interesting, from the point of view of physical applications, case in which two of the four fermionic supercharges are matrix differential operators of order 1. In this case, one of the matrix Lax–Novikov bosonic integrals plays a role of central charge of a nonlinear superalgebra, and its commutativity with first-order supercharges generates a higher-order differential equation for the superpotential that belongs to the stationary mKdV hierarchy. The second bosonic integral generates rotations between the pair of first-order supercharges and the pair of higher-order supercharges.

When the spectra of Schrödinger superpartners are different only in the lowest discrete energy level present in one of the two subsystems, which corresponds to the almost isospectral case, the superpotential has a topologically nontrivial modulated crystalline kink-type nature. This case is described by an unbroken exotic nonlinear $\mathcal{N} = 4$ supersymmetry, in which the ground state is annihilated by all four supercharges and two bosonic integrals. On the other hand, in the completely isospectral case, the pair of Schrödinger Hamiltonians is characterized

by a superpotential of a topologically trivial, modulated kink-antikink-type nature. Such pairs can be obtained from the pairs of the almost isospectral case just by sending the W-type defect associated with the lowest-energy discrete value to infinity. The completely isospectral pairs are described by a spontaneously broken exotic nonlinear $\mathcal{N}=4$ supersymmetry. Unlike the unbroken supersymmetry case, in such systems, the two states corresponding to the lowest doubly degenerate energy value are annihilated (in a generic case) only by the bosonic Lax–Novikov integrals.

When one of the two first-order supercharges is reinterpreted as the matrix Hamiltonian operator, we arrive at the Bogoliubov–de Gennes system, in which the superpotential will play the role of a scalar Dirac potential. The results presented here allow us then, particularly, to obtain new types of self-consistent condensates and associate with them new solutions for the Gross–Neveu model, which correspond to the kink- and kink-antikink-type configurations in the crystalline background. We are going to consider this problem elsewhere.

It is worth noticing that Dirac Hamiltonians with scalar potential appear, in different physical context, in the description of the low-energy charge carriers in graphene and related carbon nanostructures. This fact opens potential applications of the results in physics of condensed matter systems, following the ideas of Refs. [49–51].

The discussed constructions can be generalized to the case of the PT-symmetric one-gap potentials. To achieve this, it is sufficient to apply the complex shift considered in Sec. III to the described Hermitian systems with periodicity defects. Such systems have an immediate application in the context of the PT-symmetric quantum mechanics and optics.

An interesting development of the presented results is to "reconstruct" the time dependence for defects in a periodic background of the one-gap Lamé system in correspondence with dynamics illustrated, as an example, by Fig. 4. This would provide us a new class of solutions for the KdV and mKdV equations. At the same time, it is natural to consider the generalization of the construction to the case of quantum n-gap systems with $n>1$. One can also wonder if, somehow, both W-type and M-type defects are the result of "shrinking" bands from a more generic finite-gap Hamiltonian, under some special limit.

Finally, it would also be very interesting to look for the $(1+1)$-dimensional field theories, in which nontrivial solutions are controlled by a stability operator of the Schrödinger type [52] with the potentials of the nature considered here.

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APPENDIX: NON-SINGULARITY OF POTENTIALS

We show here that the family of Hamiltonians

$$H_{2\ell,n}=H_{0,0}-2\frac{d^2}{dx^2}(\log W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))) \tag{A1}$$

is given in terms of the *nonsingular* potentials, which correspond to the soliton defects introduced into the periodic background of the one-gap Lamé system. To achieve this, we demonstrate successively that the Wronskians appearing in the structure of $H_{0,n}$, $H_{2\ell,0}$, and, finally, $H_{2\ell,n}$ are nodeless on the real line. The notations we employ are explained in the main text.

1. Lower prohibited band

To show that the potential of $H_{0,n}$ is regular, i.e., has no zeros on the real line, we will demonstrate that

$$(-1)^{\frac{n(n+1)}{2}}W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_{n+1}}(n+1))>0. \tag{A2}$$

First, we define the two sets of functions,

$$f_n(x)\equiv(-1)^n\frac{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_{n+1}}(n+1))}{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))} \tag{A3}$$

and

$$g_n(x)\equiv(-1)^n\frac{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n),\mathcal{F}_{s_{n+2}}(n+2))}{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))}, \tag{A4}$$

which are nonphysical eigenstates of $H_{0,n}$ with eigenvalues ε^-_{n+1} and ε^-_{n+2}, respectively. We will check below that $f_n(x)>0$, while $g_n(x)$ has only one zero.

In correspondence with the definition $\mathbb{W}_{0,0}=1$ introduced in Eq. (4.21), for $n=0$, we have $f_0=\mathcal{F}_+(1)>0$, and $g_0=\mathcal{F}_-(2)$. The second function (plotted for a

particular case with $C_2 = 1$ in Fig. 3) has one zero, which we denote by x_0. Thus, we have $g_0(x) > 0$ for $x < x_0$ and $g_0(x) < 0$ for $x > x_0$.

For the case $n = 1$, we also define the functions

$$f(x) = W(\mathcal{F}_+(1), \mathcal{F}_-(2)), \qquad g(x) = W(\mathcal{F}_+(1), \mathcal{F}_+(3)), \tag{A5}$$

which appear in the numerators of (A3) and (A4). Taking into account that $\mathcal{F}$ are solutions of the stationary Schrödinger equation, it is straightforward to check that

$$f'(x) = (\varepsilon_1^- - \varepsilon_2^-)\mathcal{F}_+(1)\mathcal{F}_-(2), \tag{A6}$$

$$g'(x) = (\varepsilon_1^- - \varepsilon_3^-)\mathcal{F}_+(1)\mathcal{F}_+(3). \tag{A7}$$

As $\varepsilon_2^- < \varepsilon_1^- < 0$, we observe that $\mathrm{sign}(f'(x)) = \mathrm{sign}(\mathcal{F}_-(2))$. Then,

$$f(x_0) = \mathcal{F}_+(x_0; \beta_1^-, C_1)\mathcal{F}'_-(x_0; \beta_2^-, C_2) \tag{A8}$$

since $\mathcal{F}_-(x_0; \beta_2^-, C_2) = 0$. From the Schrödinger equation, we have also $\mathcal{F}'_-(x_0; \beta_2^-, C_2) \neq 0$, and from the definition (4.1), it follows that $\mathcal{F}'_-(x_0; \beta_2^-, C_2) < 0$. We have then $f(x_0) < 0$, and hence $\mathrm{sign}(f'(x)) = \mathrm{sign}(\mathcal{F}_-(2))$. Thus, the function $f(x)$ increases monotonically from $f(-\infty) = -\infty$, it takes a maximum negative value $f(x_0) < 0$ at $x = x_0$, and then it decreases again monotonically to $f(\infty) = -\infty$. This means that $f(x) < 0$ and, as a consequence,

$$f_1(x) = -\frac{W(\mathcal{F}_+(1), \mathcal{F}_-(2))}{\mathcal{F}_+(1)} > 0 \tag{A9}$$

for all x.

The derivative $g'(x)$ takes positive values and grows up exponentially for $x \to \pm\infty$. Therefore, $g(x)$ passes through zero only once at some point x_1. The function

$$g_1(x) = -\frac{W(\mathcal{F}_+(1), \mathcal{F}_+(3))}{\mathcal{F}_+(1)} \tag{A10}$$

has then only one zero at this point x_1 and takes positive and negative values for $x < x_1$ and $x > x_1$, respectively. So, we see that the nonphysical eigenstates f_0 and f_1 of $H_{0,0}$ and $H_{0,1}$, respectively, have no zeros, while their eigenfunctions g_0 and g_1 have one zero, where their slope is negative.

We extend now this result by induction for arbitrary n by showing that $f_n(x) > 0$ while g_n has only one zero x_n and that $g_n(x) > 0$ and $g_n(x) < 0$ for $x < x_n$ and $x > x_n$, respectively, and so, $g'_n(x_n) < 0$.

By using the Darboux–Crum construction, we can check that functions $f_n(x)$ and $g_n(x)$ are nonphysical eigenstates of the Schrödinger operator

$$H_{0,n} = H_{0,0} - 2\frac{d^2}{dx^2}\log W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n)) \tag{A11}$$

with eigenvalues ε_{n+1}^- and ε_{n+2}^-. For $n+1$, we have

$$f_{n+1}(x) = (-1)^{n+1}\frac{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+2}}(n+2))}{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1))} = -\frac{W(f_n, g_n)}{f_n}, \tag{A12}$$

$$W'(f_n, g_n) = (\varepsilon_{n+1}^- - \varepsilon_{n+2}^-)f_n g_n, \tag{A13}$$

from where we obtain that $\mathrm{sign}W'(f_n, g_n) = \mathrm{sign}g_n(x)$. The zero x_n of g_n corresponds therefore to the maximum of $W(f_n, g_n)$,

$$W(f_n, g_n)(x_n) = g'_n(x_n)f_n(x_n) < 0. \tag{A14}$$

Since $\mathrm{sign}W'(f_n, g_n) = \mathrm{sign}g_n(x)$, the function $-W(f_n, g_n)$ decreases for $x < x_n$ and increases for $x > x_n$, and then $-W(f_n, g_n)(x_n) > 0$ for all x. From Eq. (A12), we conclude that $f_{n+1}(x) > 0$ for all x.

Let us change β_{n+1}^- by β_{n+3}^- in the numerator of the function $f_n(x)$ in (A3) and redefine the resulting function as $h_n(x)$. This function takes positive values, $h_n(x) > 0$, and we obtain the following relations:

$$g_{n+1} = (-1)^{n+1}\frac{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1), \mathcal{F}_{s_{n+3}}(n+3))}{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1))} = -\frac{W(f_n, h_n)}{f_n}, \tag{A15}$$

$$W'(f_n, h_n) = (\varepsilon_{n+1}^- - \varepsilon_{n+3}^-)f_n h_n > 0. \tag{A16}$$

Consequently, $W(f_n, h_n)$ increases exponentially from $-\infty$ to $+\infty$ passing through one zero, which we call x_{n+1}. Since $f_n(x) > 0$ is a regular function, and g_{n+1} has only one zero at x_{n+1}, we find that $g_{n+1}(x) > 0$ for $x < x_{n+1}$ and $g_{n+1}(x) < 0$ for $x > x_{n+1}$.

Finally, from the definition (A3) of $f_n(x)$, we obtain

$$f_n f_{n-1} \ldots f_1 \mathcal{F}_+(1) = ((-1)^{\sum_{i=1}^n i})W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1)), \tag{A17}$$

and since

$$f_n f_{n-1}, \ldots, f_1 \mathcal{F}_+(1) > 0, \tag{A18}$$

we demonstrate the necessary relation (A2).

2. Upper prohibited band

To show that $H_{2\ell,0}$ is nonsingular on the whole real line, we show that the Wronskian is a regular nodeless function

$W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell))$, where the functions $\Phi_+(2l-1)$ and $\Phi_-(2l)$, $l=1,2\ldots$ correspond to a generalization of those defined in (4.30) and (4.31) for $\ell=1$.

Before, we showed that $W(\Phi_+(1),\Phi_-(2))<0$ by choosing parameters $0<\beta_1^+<\beta_2^+<\mathbf{K}$. This condition means that $1>\varepsilon_1^+>\varepsilon_2^+>k'^2$ for the eigenvalues of the nonphysical eigenstates $\Phi_+(1)$ and $\Phi_-(2)$ inside the intermediate forbidden band of $H_{0,0}$.

To demonstrate the validity of the formulated statement for the next case $\ell=2$, we define an eigenstate of the one-gap Lamé system with the displaced argument, $x\to x+\beta_3^++i\mathbf{K}'$, in the following form:

$$\breve{\Phi}[1](x,\beta_3^+)=\frac{W(\Psi_+^{\beta_3^+}(x),\Phi_+(1))}{\Psi_+^{\beta_3^+}(x)}. \tag{A19}$$

This state has an infinite number of poles at the zeros of $\Psi_+^{\beta_3^+}(x)$. Between each pair of poles, $\breve{\Phi}[1](x,\beta_3^+)$ does not change the sign and takes nonzero values. Its sign is inverted in the neighbor regions separated by poles. From the theorem on zeros, the linearly independent state

$$\breve{\Phi}[2](x;\beta_3^+)=\frac{W(\Psi_+^{\beta_3^+}(x),\Phi_-(2))}{\Psi_+^{\beta_3^+}(x)} \tag{A20}$$

has also an infinite number of poles, but between each pair of poles, it possesses one zero, which we denote as x_i. The function (A20) preserves the sign when the argument passes through any pole.

Now, it is necessary to show that $W(\breve{\Phi}[1],\breve{\Phi}[2])$ does not have zeros. For this, we redefine the function $\breve{\Phi}[2]$ up to a sign in such a way that its derivative in some x_{i_0} will be positive. In the same way, we also redefine, up to a global sign, the function $\breve{\Phi}[1](x)$ to have $\breve{\Phi}[1](x_{i_0})<0$. Thus, we obtain that

$$W(\breve{\Phi}[1],\breve{\Phi}[2])(x_i)=\breve{\Phi}[1](x_i)\breve{\Phi}'[2](x_i)<0, \tag{A21}$$

while

$$W'(\breve{\Phi}[1],\breve{\Phi}[2])=(\varepsilon_1^+-\varepsilon_2^+)\breve{\Phi}[1]\breve{\Phi}[2]. \tag{A22}$$

The function $W(\breve{\Phi}[1],\breve{\Phi}[2])$ has a local extremum at each x_i, and its derivative is positive for $x<x_i$ until a pole and is negative for $x>x_i$ until the next pole since x_i is a local maximum of $W(\breve{\Phi}[1],\breve{\Phi}[2])(x)$. From here, we conclude that $W(\breve{\Phi}[1],\breve{\Phi}[2])(x)$ does not have zeros and hence is of one sign.

Because of the identity

$$W(\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(x))=\Psi_+^{\beta_3^+}(x)W(\breve{\Phi}[1],\breve{\Phi}[2])(x;\beta_3^+), \tag{A23}$$

the Wronskian $W(\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(\pm x))$ has exactly the same zeros as $\Psi_\pm^{\beta_3^+}(x)$. Note that we have $W(\Phi_+(-x;\beta_1^+,1/C_1),\ \Phi_-(-x;\beta_2^+,1/C_2),\Psi_+^{\beta_3^+}(-x))=-W(-\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(-x))=-W(\Phi_+(1),\Phi_-(2),-\Psi_+^{\beta_3^+}(-x))$. Using the Wronskian properties, it is easy to see that $W(a,b)(x)=-W(a,b)(-x)$ and $W(a,b,c)(x)=-W(a,b,c)(-x)$, but $W(a,b,c,d)(x)=W(a,b,c,d)(-x)$. Taking in account the above relations, we can write

$$\mathrm{sign}W(\breve{\Phi}[1],\breve{\Phi}[2])(x;\beta_3^+)=\mathrm{sign}W(\breve{\Phi}[1],\breve{\Phi}[2])(-x;\beta_3^+). \tag{A24}$$

Thus, the zeros of the nonphysical states of $H_{2,0}$,

$$\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3))}{W(\Phi_+(1),\Phi_-(2))}\quad\text{and}\quad\frac{W(\Phi_+(1),\Phi_-(2),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))}, \tag{A25}$$

are within the intervals $\mathcal{I}_n^+(\beta_3^+)$ and $\mathcal{I}_n^-(\beta_4^+)$, respectively, see Eq. (4.34), where $\mathcal{I}_n^+(3)\bigcap\mathcal{I}_n^-(4)=\emptyset$. As a consequence of the theorem on zeros, their zeros are alternated.

Next, we can check that under the condition $0<\beta_1^+<\beta_2^+<\beta_3^+<\beta_4^+<\mathbf{K}$, the Wronskian

$$W\left(\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3))}{W(\Phi_+(1),\Phi_-(2))},\frac{W(\Phi_+(1),\Phi_-(2),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))}\right)=\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))} \tag{A26}$$

does not have zeros nor the function $W(\Phi_+(1),\Phi_-(2),\Phi_+(3),\Phi_-(4))$.

This result can be generalized for the case of the Wronskian of 2ℓ states, $W(\Phi_+(1),\Psi_-(2),\ldots,\Phi_+(2\ell-1),\Phi_-(2\ell))$, under the condition $0<\beta_1^+<\beta_2^+<\ldots<\beta_{2\ell}^+<\mathbf{K}$.

Using the identity

$$\begin{aligned}&W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Psi_+^{\beta_{2\ell+1}^+}(x))\\&=W(\Phi_+(1),\ldots,\Phi_-(2\ell-2),\Psi_+^{\beta_{2\ell+1}^+}(x))\\&\quad\times W(\breve{\Phi}[1,\ldots,2\ell-1],\breve{\Phi}[1,\ldots,2\ell-2,2\ell]),\end{aligned} \tag{A27}$$

we have

$$\begin{aligned}&\Psi_+^{\beta_{2\ell+1}^+}(x)W(\breve{\Phi}[1],\breve{\Phi}[2])\times W(\breve{\Phi}[1,2,3],\breve{\Phi}[1,2,4])\times\ldots\\&\quad\times W(\breve{\Phi}[1,\ldots,2\ell-2,2\ell-1],\breve{\Phi}[1,\ldots,2\ell-2,2\ell])\\&=W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Psi_+^{\beta_{2\ell+1}^+}(x)),\end{aligned} \tag{A28}$$

where

$$\breve{\Phi}[1,\ldots,l,l+r](x,\beta^+) = \frac{W(\Psi_+^{\beta^+}(x),\Phi_+(1),\ldots,\Phi_-(2l),\Phi_{s_{2l+r}}(2l+r))}{W(\Psi_+^{\beta^+}(x),\Phi_+(1),\ldots,\Phi_-(2l))} \tag{A29}$$

and $r=1,2,\ l=0,1,\ldots$. Having in mind all previous demonstrations, it is clear that

$$|W(\breve{\Phi}[1,\ldots,2l-2,2l-1],\breve{\Phi}[1,\ldots,2l-2,2l])|>0, \tag{A30}$$

and the functions

$$\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_+(2\ell+1))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \quad \text{and} \quad \frac{W(\Phi_+(1),\ldots,.\Phi_-(2\ell),\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \tag{A31}$$

have alternating zeros in the intervals $\mathcal{I}_n^+(\beta_{2\ell+1}^+)$ and $\mathcal{I}_n^-(\beta_{2\ell+2}^+)$, respectively. Then,

$$W\left(\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_+(2\ell+1))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))},\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}\right)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \tag{A32}$$

is regular and has no zeros, which means that $W(\Phi_+(1),\ldots,\Phi_-(2\ell+2))$ is nonsinguar and nodeless if and only if $W(\Phi_+(1),\ldots,\Phi_-(2\ell))$ is regular and has no zeros.

Besides, if the potentials of the systems $H_{2\ell,0}$ are nonsingular for all real x, by taking limits $C_l\to\infty$ or $C_l\to 0$, the regularity is preserved, and we get a regular Hamiltonians $H_{2\ell-1,0}$ with $2\ell-1$ states in the gap of the Lamé system.

3. Mixed case

Finally, using the all previous demonstrations, we show that the most general Hamiltonian

$$H_{2\ell,n}=H_{0,0}-2\frac{d^2}{dx^2}(\log W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))) \tag{A33}$$

has also a nonsingular potential. To this aim, we define

$$F_{2\ell}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),F(x;\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}, \tag{A34}$$

which is a nonphysical eigenstate of $H_{2\ell,0}$ with eigenvalue $\mathcal{E}(\beta^-+i\mathbf{K}')$. Using the Wronskian identity

$$\begin{aligned} W(\tilde{\Phi}_1,\ldots,\tilde{\Phi}_l) &= W(W(F,\Phi_1)/F,\ldots,W(F,\Phi_l)/F) \\ &= W(F,\Phi_1,\ldots,\Phi_l)/F, \end{aligned} \tag{A35}$$

where $\tilde{\Phi}=W(F,\Phi)/F$, we obtain

$$\begin{aligned} F_{2\ell}(x;\beta^-) &= \frac{W(\tilde{\Phi}_+(1),\ldots,\tilde{\Phi}_-(2\ell))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}F(x;\beta^-) \\ &= G_{2\ell}(x;\beta^-)F(x;\beta^-). \end{aligned} \tag{A36}$$

$\tilde{\Phi}_i$ is the eigenstate of the displaced Lamé system $H_{0,0}(x+\beta^-)$, with the properties similar to those as Φ_i. We have shown that $W(\Phi_+(1),\ldots,\Phi_-(2\ell))$ is nodeless and takes finite values of a definite sign. This implies that $W(\tilde{\Phi}_+(1),\ldots,\tilde{\Phi}_-(2\ell))$ share the same properties. Hence, function $G_{2\ell}(x;\beta^-)$ also possesses the same indicated properties. Taking into account the properties of the functions inside the Wronskian under the reflection $x\to -x$, it is not difficult to show that $\mathrm{sign}G_{2\ell}(x;\beta^-)=\mathrm{sign}G_{2\ell}(x;-\beta^-)$. Having the identity $F(-x;\beta^-)=F(x;-\beta)$, we find that

$$\mathcal{F}_{2\ell,+}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\mathcal{F}_+(\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2l))} \tag{A37}$$

$$= CG_{2\ell}(x;\beta^-)F(x;\beta^-) + \frac{1}{C}G_{2\ell}(x;-\beta^-)F(-x;\beta^-). \tag{A38}$$

Since $G_{2\ell}(x;\pm\beta^-)$ take values of the same sign and increase exponentially, the function $\mathcal{F}_{2\ell,+}$ has no zeros. Then,

$$\mathcal{F}_{2\ell,-}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\mathcal{F}_-(\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \tag{A39}$$

$$= CG_{2\ell}(x;\beta^-)F(x;\beta^-) - \frac{1}{C}G_{2\ell}(x;-\beta^-)F(-x;\beta^-) \tag{A40}$$

has only one zero. Here, the functions $\mathcal{F}_{2\ell,\pm}$ are linearly independent eigenstates of the operator $H_{2\ell,0}$ with eigenvalues $\mathcal{E}(\beta^-+i\mathbf{K}')$, which are analogous to the eigenfunctions $\mathcal{F}_\pm$ of the Lamé system $H_{0,0}$; see (4.1). Using the arguments presented in Appendix A 1, one can show that

$$W(\mathcal{F}_{2\ell,+}(1),\ldots,\mathcal{F}_{2\ell,s_{n+1}}(n+1)) \tag{A41}$$

has no zeros. From the Crum theorem,

$$H_{2\ell,n} = H_{2\ell,0} - 2\frac{d^2}{dx^2}\log W(\mathcal{F}_{2\ell,+}(1),\ldots,\mathcal{F}_{2\ell,s_n}(n)) = H_{0,0} - 2\frac{d^2}{dx^2}\log \mathbb{W}_{2\ell,n}, \quad \text{(A42)}$$

and it follows that

$$W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n)) \quad \text{(A43)}$$

is a smooth and nodeless function.

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Capítulo 4

Supersimetría extendida $N = 4$ en mecánica cuántica con coeficientes de estructura nolineales

Como hemos visto en los capítulos anteriores la transformación de Crum-Darboux nos provee de una formulacón nolineal de supersimetría en mecánica cuántica, pudiendose interpretar el Hamiltoniano supersimétrico como el de una partícula no relativista de espín $\frac{1}{2}$, si sumamos a esto que el par de potenciales supersimétricos son transformaciones de Darboux de un potencial transparente($g = 0$) o finite(g)-gap $u_{g,l}$ y por tanto solución de alguna ecuación de la jerarquía inhomogénea de KdV estacionaria, entonces dicho Hamiltoniano supersimétrico tendrá una integral adicional análoga a la integral de Lax-Novikov asociada a $H_{g,l}$ ($P_{2g+2l+1}$), definidos en 1.2, en adelante mostraremos como la supersimetría usual $N = 2$ (con dos integrales fermiónicas) se ve extendida a una supersimetría $N = 4$ con coeficientes de estructura nolineales.

Sea $\mathcal{H} = \mathrm{diag}(H_{g,l}, H'_{g',l'})$ un sistema extendido completamente transparente o finite-gap (con g y g' el genero de la curva hiper-elíptica asociada al espectro del Hamiltoniano $H_{g,l}$ y $H'_{g',l'}$, respectivamente) y que por lo tanto posee una integral de Lax-Novikov extendida en la forma $\mathcal{P}_1 = \mathrm{diag}(P_{2g+2l+1}, P'_{2g'+2l'+1})$ y una integral (de graduación) trivial σ_3, acá los datos de dispersión del Hamiltoniano $H_{g,l}$ están dados por $\mathbf{v}, \boldsymbol{\phi} \in \mathbb{C}^g$, la matriz τ ($g \times g$ con $\mathrm{Im}\tau > 0$), $a_{j,b}$ y $r_{j,b}$, $j = 1, \ldots, l$ y los datos espectrales del Hamiltoniano $H'_{g',l'}$ están dados por $\mathbf{v}', \boldsymbol{\phi}' \in \mathbb{C}^{g'}$, la matriz τ' ($g' \times g'$con $\mathrm{Im}\tau' > 0$), $a'_{j',b}$ y $r'_{j',b}$, $j' = 1, \ldots, l'$, $a, b = 1, 2$.

Para estudiar las posibles simetrías del Hamiltoniano $\mathcal{H}$ debemos diferenciar entre tres casos i) sí $g' \neq g$ o $u'_{g',0} \neq \tilde{u}^{r_1,r_2,\ldots,r_n}_{g,0}$, $\tilde{u}^{r_1,r_2,\ldots,r_n}_{g,0}$ es alguna transformación de tipo auto Darboux de orden n de $u_{g,0}$ (Una transformacón de Crum-Darboux compuesta de n auto transformaciones de Darboux iteradas), con $n \leq g$: en este caso $H_{g,l}$ y $H'_{g',l'}$ no pueden ser conectados mediante transformaciones de Darboux, ya que estas no pueden modificar el espectro de las bandas y por otro lado aunque ambos operadores sean isospectrales es posible que no puedan ser conectados a traves de efectos de dispersión de un número finito de solitones (transformaciones de tipo auto

Darboux), el operador de graduación $\Gamma_1 = \sigma_3$ define $\mathcal{P}$ como integral bosónica con álgebra

$$[\mathcal{H}, \mathcal{P}_b] = 0, \quad b = 1, 2, \quad \mathcal{P}_2 = \sigma_3 \mathcal{P}_1, \tag{4.1}$$

pero el operador de graduación nolocal $\Gamma_2 = \Gamma$, $\Gamma O = (-1)^{|O|} O\Gamma$, $\Gamma^{\#} = \Gamma$, $\Gamma^2 = 1$ define $\mathcal{P}_1$ y $\mathcal{P}_2 = \Gamma\mathcal{P}_1$ como integrales fermiónicas ($\{\Gamma, \mathcal{P}_b\} = 0$) cuya superálgebra toma la forma

$$[\mathcal{H}, \mathcal{P}_a] = 0, \tag{4.2}$$

$$\{\mathcal{P}_a, \mathcal{P}_b\} = \delta_{ab} \left((\mathbf{1} + \sigma_3) \prod_{j=0}^{2g} (\mathcal{H} - E_j) \prod_{j=1}^{l} (\mathcal{H} - z_j)^2 + (\mathbf{1} - \sigma_3) \prod_{j'=0}^{2g'} (\mathcal{H} - E'_{j'}) \prod_{j'=1}^{l'} (\mathcal{H} - z'_{j'})^2 \right), \tag{4.3}$$

la cual muestra una reducción espontánea de orden en los limites mostrados en la sección anterior, $\Gamma_3 = \Gamma\sigma_3$ reproduce el mismo superálgebra.

Emerge un superálgebra extendida con coeficientes nolineales en dos casos ii) cuando $u'_{g',0} = u_{g,0}$ y iii) cuando $u'_{g',0} = \tilde{u}^{r_1,r_2,\ldots,r_n}_{g,0}$: en ambos casos $H_{g,l}$ y $H'_{g',l'}$ están conectados vía transformaciones de Crum-Darboux. En adelante estudiaremos por separado el superálgebra de Lie de los generadores de simetrías en cada uno de estos casos.

4.1. Supersimetría extedida con coeficientes de estructura nolineales, en ausencia de auto transformaciones de Darboux.

En el caso ii) $\mathbb{A}'_{l'}$ entrelaza $H_{g,0} = H'_{g,0}$ con $H'_{g,l'}$[1] mientras que $\mathbb{A}_l$ entrelaza $H_{g,0} = H'_{g,0}$ con $H_{g,l}$, por lo que $\mathbb{Y}_{l'+l} = \mathbb{A}'_{l'}\mathbb{A}^{\#}_l$ y $\mathbb{X}_{2g+l+l'+1} = \mathbb{A}'_{l'}P_{2g+1}(u_{g,0}, \partial\sigma(H_{g,0}))\mathbb{A}^{\#}_l$ entrelazan $H_{g,l}$ con $H'_{g,l'}$ de la forma

$$\mathbb{Y}_{l+l'}H_{g,l} = H'_{g,l'}\mathbb{Y}_{l+l'}, \quad \mathbb{Y}^{\#}_{l+l'}H'_{g,l'} = H_{g,l}\mathbb{Y}^{\#}_{2g+l+l'+1}, \tag{4.4}$$

$$\mathbb{X}_{2g+l+l'+1}H_{g,l} = H'_{g,l'}\mathbb{X}_{2g+l+l'+1}, \quad \mathbb{X}^{\#}_{2g+l+l'+1}H'_{g,l'} = H_{g,l}\mathbb{X}^{\#}_{2g+l+l'+1}, \tag{4.5}$$

estos operadores de entrelazamiento definen las siguientes factorizaciones nolineales

$$\mathbb{Y}^{\#}_{l+l'}\mathbb{Y}_{l+l'} = \prod_{j=1}^{l}(H_{g,l} - z_j)^2 \prod_{j=1}^{l'}(H_{g,l} - z'_j)^2, \tag{4.6}$$

$$\mathbb{Y}_{l+l'}\mathbb{Y}^{\#}_{l+l'} = \prod_{j=1}^{l}(H'_{g,l'} - z_j)^2 \prod_{j=1}^{l'}(H'_{g,l'} - z'_j)^2, \tag{4.7}$$

$$\mathbb{X}^{\#}_{2g+l+l'+1}\mathbb{X}_{2g+l+l'+1} = \prod_{j=0}^{2g}(H_{g,l} - E_j) \prod_{j=1}^{l}(H_{g,l} - z_j)^2 \prod_{j=1}^{l'}(H_{g,l} - z'_j)^2, \tag{4.8}$$

[1]Los operadores primados solo indican que han sido construidos con datos de dispersión distindos, en este caso el número de estados l' puede ser distinto de l y los estados que se ocupan para construir $H'_{g,l'}$ y $\mathbb{A}'_{l'}$ pueden coincidir o no con los usados para construir $H_{g,l}$ y $\mathbb{A}_l$ según se indique en el texto.

$$\mathbb{X}_{2g+l+l'+1}\mathbb{X}^{\#}_{2g+l+l'+1} = \prod_{j=0}^{2g}(H'_{g,l'} - E_j)\prod_{j=1}^{l}(H'_{g,l'} - z_j)^2\prod_{j=1}^{l'}(H'_{g,l'} - z'_j)^2, \tag{4.9}$$

y para $z_i \neq z'_j$ nos permite construir doce integrales de movimiento irreducibles para $\mathcal{H}$

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & \mathbb{Y}^{\#}_{l+l'} \\ \mathbb{Y}_{l+l'} & 0 \end{pmatrix}, \quad \mathcal{Q}_2 = i\sigma_3\mathcal{Q}_1, \quad \mathcal{Q}_3 = \Gamma i^{l+l'}\mathcal{Q}_1, \quad \mathcal{Q}_4 = i^{l+l'+1}\sigma_3\Gamma\mathcal{Q}_1, \tag{4.10}$$

$$\mathcal{S}_1 = \begin{pmatrix} 0 & \mathbb{X}2g + l' + l + 1^{\#} \\ \mathbb{X}_{2g+l'+l+1} & 0 \end{pmatrix}, \quad \mathcal{S}_2 = i\sigma_3\mathcal{S}_1, \quad \mathcal{S}_3 = i^{l+l'+1}\Gamma\mathcal{S}_1, \quad \mathcal{S}_4 = i^{l+l'}\sigma_3\Gamma\mathcal{S}_1, \tag{4.11}$$

$$\mathcal{P}_1 = \begin{pmatrix} P_{2g+2l+1} & 0 \\ 0 & P'_{2g+2l'+1} \end{pmatrix}, \quad \mathcal{P}_2 = \sigma_3\mathcal{P}_1, \quad \mathcal{P}_3 = i\Gamma\mathcal{P}_1, \quad \mathcal{P}_2 = i\sigma_3\Gamma\mathcal{P}_1\,. \tag{4.12}$$

junto con tres posibles operadores de graduación (dos de ellos no locales) $\Gamma_i = \sigma_3, \Gamma, \Gamma\sigma_3$ $[\Gamma_i, \mathcal{H}] = 0$, $\Gamma_i^2 = 1$

Γ_i| IOM	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
σ_3	-	-	-	-	-	-	-	-	+	+	+	+
Γ	+	+	+	+	-	-	-	-	-	-	-	-
$\sigma_3\Gamma$	-	-	-	-	+	+	+	+	-	-	-	-

Cuadro 4.1: Representación de la graduación de las integrales de movimiento para los diferentes operadores de graduacón, $+$ para operadores bosónicos y $-$ para operadores fermiónicos.

A modo de ejemplo el subsuperálgebra sin el operador de graduación no local Γ y σ_3 operador de graduación, presenta cuatro integrales fermiónicas antidiagonales: $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{S}_1$ y $\mathcal{S}_2$,las cuales anticonmutan con σ_3 más dos integrales bosónicas diagonales: $\mathcal{P}_1$ y $\mathcal{P}_2$, las cuales conmutan con σ_3. Su subsuperálgebra toma la forma

$$[\mathcal{H}, \mathcal{Q}_a] = [\mathcal{H}, \mathcal{S}_a] = [\mathcal{H}, \mathcal{P}_a] = [\mathcal{H}, \sigma_3] = [\mathcal{P}_a, \mathcal{P}_b] = 0, \quad a, b = 1, 2, \tag{4.13}$$

$$\{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{a,b}\prod_{j=1}^{l}(\mathcal{H} - z_j)^2\prod_{j=1}^{l'}(\mathcal{H} - z'_j)^2, \tag{4.14}$$

$$\{\mathcal{S}_a, \mathcal{S}_b\} = 2\delta_{a,b}\prod_{j=0}^{2g}(\mathcal{H} - E_j)\prod_{j=1}^{l}(\mathcal{H} - z_j)^2\prod_{j=1}^{l'}(\mathcal{H} - z'_j)^2, \tag{4.15}$$

$$\{\mathcal{S}_a, \mathcal{Q}_b\} = \delta_{a,b}\left(\left(\prod_{j=1}^{l'}(\mathcal{H} - z'_j) + \prod_{j=1}^{l}(\mathcal{H} - z_j)\right)\mathcal{P}_1 + \left(\prod_{j=1}^{l'}(\mathcal{H} - z'_j) - \prod_{j=1}^{l}(\mathcal{H} - z_j)\right)\mathcal{P}_2\right), \tag{4.16}$$

$$[\mathcal{P}_1, \mathcal{Q}_a] = (-1)^a i\left(\prod_{j=1}^{l}(\mathcal{H} - z_j) - \prod_{j=1}^{l'}(\mathcal{H} - z'_j)\right)\mathcal{S}_{a'}, \tag{4.17}$$

$$[\mathcal{P}_2, \mathcal{Q}_a] = (-1)^a i \left(\prod_{j=1}^{l} (\mathcal{H} - z_j) + \prod_{j=1}^{l'} (\mathcal{H} - z'_j) \right) \mathcal{S}_{a'}, \tag{4.18}$$

$$[\mathcal{P}_1, \mathcal{S}_a] = (-1)^a i \prod_{j=0}^{2g} (\mathcal{H} - E_j) \left(\prod_{j=1}^{l} (\mathcal{H} - z_j) - \prod_{j=1}^{l'} (\mathcal{H} - z'_j) \right) \mathcal{Q}_{a'}, \tag{4.19}$$

$$[\mathcal{P}_2, \mathcal{S}_a] = (-1)^a i \prod_{j=0}^{2g} (\mathcal{H} - E_j) \left(\prod_{j=1}^{l} (\mathcal{H} - z_j) + \prod_{j=1}^{l'} (\mathcal{H} - z'_j) \right) \mathcal{Q}_{a'}, \tag{4.20}$$

acá $a, a' = 1, 2$, $a \neq a'$. Los operadores $\mathcal{P}$ juegan el rol de generadores de rotaciones entre las integrales $\mathcal{Q}$ y $\mathcal{S}$, pero además en el caso completamente isospectral, $l = l'$ y $z_i = z'_i$, $\mathcal{P}_1$ juega un rol más importante como carga central. Este último resultado como veremos más adelante conectara la jerarquía de KdV con la de mKdV.

Esta supersimetría no esta rota si el estado fsico de energía más baja entre los estados de $H_{g,l}$ y $H'_{g,l'}$ es un estado ligado en cambio esta parcialmente rota si el estado más bajo corresponde a un borde de banda ($\mathcal{Q}$).

Este sperálgebra muestra una reducción de orden espontánea cuando los estados usados en la Crum-Darboux construcción de los Hamiltonianos coinciden en la forma $\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x) = \psi_{a'_{j',1},a'_{j',2}}(r'_{j',1}, r'_{j',2}, x)$ acá $j = 1, 2, \ldots, l$ y $j' = 1, 2, \ldots, l'$, Sin perder generalidad[2] podemos estudiar el caso en que $\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x) = \psi_{a'_{j,1},a'_{j,2}}(r'_{j,1}, r'_{j,2}, x)$ para $j = 1, 2, \ldots, k$ en este caso el par $(H_{g,l}, H'_{g,l'})$ están ambos crum-Darboux conectados con $H_{g,k}$. En este caso el operador de entrelazamiento $\mathbb{A}'_{l'}\mathbb{A}_l^{\#}$ contiene información redundante[3], en este caso es $\mathbb{Y}_{l+l',k} = \mathbb{A}_{j,k+1}\mathbb{A}'_{j,k+1}{}^{\#}$el constituyente irreducible en $\mathbb{Y}_{l+l'}$ el cual se factoriza en la forma $\mathbb{Y}_{l+l'} = \mathbb{Y}_{l+l',k} \prod_{j=1}^{k} (H_{g,l} - z_j)$ por lo que la integral $\mathcal{Q}_a$ debe ser cambiada por una integral reducida $\mathcal{Q}_{a,k}$, tal que $\mathcal{Q}_a = \mathcal{Q}_{a,k} \prod_{j=1}^{k} (\mathcal{H} - z_j)$

$$\mathcal{Q}_{1,k} = \begin{pmatrix} 0 & \mathbb{Y}^{\#}_{l+l',k} \\ \mathbb{Y}_{l+l',k} & 0 \end{pmatrix}, \quad \mathcal{Q}_{2,k} = i\sigma_3 \mathcal{Q}_{1,k}, \quad \mathcal{Q}_{3,k} = \Gamma \mathcal{Q}_{1,k}, \quad \mathcal{Q}_{4,k} = i\sigma_3 \Gamma \mathcal{Q}_{1,k}. \tag{4.21}$$

Las demás integrales no se ven reducidas en este limite y el subsuperálgebra reducida toma la siguiente forma

$$[\mathcal{H}, \mathcal{Q}_{a,k}] = 0, \quad a = 1, 2, \tag{4.22}$$

$$\{\mathcal{Q}_{a,k}, \mathcal{Q}_{b,k}\} = 2\delta_{a,b} \prod_{j=k+1}^{l} (\mathcal{H} - z_j)^2 \prod_{j=k+1}^{l'} (\mathcal{H} - z'_j)^2, \tag{4.23}$$

[2] Siempre es posible una redefinición en el índice $j \to n(j)$, $j' \to n'(j')$, con n y n' una permutación de $(1, 2, \ldots, l)$ y $(1, 2, \ldots, l')$ respectivamente.

[3] Guarda relación con los arboles genealógicos por ejemplo si dos personas tienen un padre en común ($H_{g,k}$) es obvio que tienen un abuelo en común ($H_{g,k-1}$) y asi hacia atras hasta llegar a un tatara^{k-2}abuelo en común ($H_{g,0}$).

$$\{\mathcal{S}_a, \mathcal{Q}_{b,k}\} = \delta_{a,b}\left(\left(\prod_{j=k+1}^{l'}(\mathcal{H}-z'_j)+\prod_{j=k+1}^{l}(\mathcal{H}-z_j)\right)\mathcal{P}_1+\left(\prod_{j=k+1}^{l'}(\mathcal{H}-z'_j)-\prod_{j=k+1}^{l}(\mathcal{H}-z_j)\right)\mathcal{P}_2\right), \tag{4.24}$$

$$[\mathcal{P}_1, \mathcal{Q}_{a,k}] = (-1)^a i\left(\prod_{j=k+1}^{l}(\mathcal{H}-z_j)-\prod_{j=k+1}^{l'}(\mathcal{H}-z'_j)\right)\mathcal{S}_{a'}, \tag{4.25}$$

$$[\mathcal{P}_2, \mathcal{Q}_{a,k}] = (-1)^a i\left(\prod_{j=k+1}^{l}(\mathcal{H}-z_j)+\prod_{j=k+1}^{l'}(\mathcal{H}-z'_j)\right)\mathcal{S}_{a'}, \tag{4.26}$$

$$[\mathcal{P}_1, \mathcal{S}_a] = (-1)^a i\prod_{j=0}^{2g}(\mathcal{H}-E_j)\prod_{j=1}^{k}(\mathcal{H}-z_j)\left(\prod_{j=1}^{l}(\mathcal{H}-z_j)-\prod_{j=1}^{l'}(\mathcal{H}-z'_j)\right)\mathcal{Q}_{a',k}, \tag{4.27}$$

$$[\mathcal{P}_2, \mathcal{S}_a] = (-1)^a i\prod_{j=0}^{2g}(\mathcal{H}-E_j)\prod_{j=1}^{k}(\mathcal{H}-z_j)\left(\prod_{j=1}^{l}(\mathcal{H}-z_j)+\prod_{j=1}^{l'}(\mathcal{H}-z'_j)\right)\mathcal{Q}_{a',k}. \tag{4.28}$$

Es posible notar que el orden de la no linealidad de los coeficientes de estructura decrece cuando existen estas coincidencias entre los estados generadores de la transformación de Crum-Darboux. Es necesario notar que en que $H_{g,l}$ y $H'_{g,l'}$ son completamente isospectrales ($\sigma(H_{g,l}) = \sigma(H'_{g,l})$, $l = l'$) la integral $\mathcal{P}_1$ juega el rol de elemento central del superálgebra. Tambien es posible notar como las super cargas fermiónicas factorizan a la integrales bosónicas (4.24).

4.2. Supersimetría extedida con coeficientes de estructura nolineales, en presencia de auto transformaciones de Darboux.

Definamos el Hamiltoniano [4] como no singular y real o $\mathcal{PT}$ simétrico

$$\breve{H}'^{k}_{g,l'} = \breve{H}^{k}_{g,0} - 2\frac{d^2}{dx^2}\ln(\mathbb{W}^{\mathcal{D}_k}_{l'}(\mathcal{D}_k\psi_{a'_{1,1},a'_{1,2}}(r'_{1,1}, r'_{1,2}, x), \ldots, \mathcal{D}_k\psi_{a'_{l',1},a'_{l',2}}(r_{l',1}, r'_{l',2}, x))) \tag{4.29}$$

con

$$\mathbb{W}^{\mathcal{D}_k}_{j} = \mathbb{W}(\mathcal{D}_k\psi_{a'_{1,1},a'_{1,2}}(r'_{1,1}, r'_{1,2}, x), \ldots, \mathcal{D}_k\psi_{a'_{j,1},a'_{j,2}}(r_{j,1}, r'_{j,2}, x))), \tag{4.30}$$

el operador $\mathcal{D}_k$ es definido como en (2.11). El Hamiltoniano definido en (4.29) está Crum-Darboux conectado con $\breve{H}^{k}_{g,0}$ por medio del operado de entrelazamiento

$$\mathbb{A}'^{\mathcal{D}_k}_{l'} = A'^{\mathcal{D}_k}_{l'} \times \cdots \times A'^{\mathcal{D}_k}_{1}, \quad A'^{\mathcal{D}_k}_{j} = \frac{\mathbb{W}^{\mathcal{D}_k}_{j}}{\mathbb{W}^{\mathcal{D}_k}_{j-1}}\frac{d}{dx}\frac{\mathbb{W}^{\mathcal{D}_k}_{j-1}}{\mathbb{W}^{\mathcal{D}_k}_{j}} \tag{4.31}$$

en la forma $\mathbb{A}'^{\mathcal{D}_k}_{l'}\breve{H}^{k}_{g,0} = \breve{H}'^{k}_{g,l'}\mathbb{A}'^{\mathcal{D}_k}_{l'}$ y por lo tanto $H_{g,l}$ y $\breve{H}'^{k}_{g,l'}$ están conectados mediante una mezcla de auto transformaciones de Crum-Darbux y transformaciones solitónicas de tipo Crum-Darboux, constituyendo operadores de entrelazamiento irreducibles entre $H_{g,l}$ y $\breve{H}'^{k}_{g,l'}$ en la forma

$$\mathcal{D}_{l',k,l} = \mathbb{A}'^{\mathcal{D}_k}_{l'}\mathcal{D}_k\mathbb{A}^{\#}_{l}, \quad \mathcal{B}_{l',2g+1-k,l} = \mathbb{A}'^{\mathcal{D}_k}_{l'}\mathcal{B}_{2g+1-k}\mathbb{A}^{\#}_{l} \tag{4.32}$$

[4] $\breve{H}'^{k}_{g,l'}$ podría ser singular

$\mathcal{B}_{2g+1-k}$ es definido en (2.12). Los operadores de intrelazamiento en (4.32) definen integrales de movimiento antidiagonales para $\breve{\mathcal{H}} = \mathrm{diag}(H_{g,l}, \breve{H'}^{k}_{g,l'})$ que denotaremos por $\breve{\mathcal{Q}}$ y $\breve{\mathcal{S}}$ y están dadas en la forma

$$\breve{\mathcal{Q}}_1 = \begin{pmatrix} 0 & \mathcal{D}^{\#}_{l',k,l} \\ \mathcal{D}_{l',k,l} & 0 \end{pmatrix}, \quad \breve{\mathcal{Q}}_2 = i\sigma_3\breve{\mathcal{Q}}_1, \quad \breve{\mathcal{Q}}_3 = i^{l+l'+k}\Gamma\breve{\mathcal{Q}}_1, \quad \breve{\mathcal{Q}}_4 = i^{l+l'+k+1}\sigma_3\Gamma\breve{\mathcal{Q}}_1, \tag{4.33}$$

$$\breve{\mathcal{S}}_1 = \begin{pmatrix} 0 & \mathcal{B}^{\#}_{l',2g+1-k,l} \\ \mathcal{B}_{l',2g+1-k,l} & 0 \end{pmatrix}, \quad \breve{\mathcal{S}}_2 = i\sigma_3\breve{\mathcal{S}}_1, \quad \breve{\mathcal{S}}_3 = i^{l+l'+1+k}\Gamma\breve{\mathcal{S}}_1, \quad \breve{\mathcal{S}}_4 = i^{l+l'+k}\sigma_3\Gamma\breve{\mathcal{S}}_1, \tag{4.34}$$

además de las respectivas integrales de Lax-Novikov

$$\breve{\mathcal{P}}_1 = \begin{pmatrix} P_{2g+2l+1}(u_{g,l}, \partial\sigma(H_{g,l})) & 0 \\ 0 & P_{2g+2l'+1}(\breve{u'}^{\mathcal{D}_k}_{g,l'}, \partial\sigma(\breve{H'}^{\mathcal{D}_k}_{g,l'})) \end{pmatrix}, \tag{4.35}$$

$$\breve{\mathcal{P}}_2 = \sigma_3\breve{\mathcal{P}}_1, \quad \breve{\mathcal{P}}_3 = i\Gamma\breve{\mathcal{P}}_1, \quad \breve{\mathcal{P}}_2 = i\sigma_3\Gamma\breve{\mathcal{P}}_1\,. \tag{4.36}$$

las cuales describen el siguiente superálgebra para σ_3 como operador de graduación.

$$[\breve{\mathcal{H}}, \breve{\mathcal{Q}}_a] = [\breve{\mathcal{H}}, \breve{\mathcal{S}}_a] = [\breve{\mathcal{H}}, \breve{\mathcal{P}}_a] = [\breve{\mathcal{H}}, \sigma_3] = [\breve{\mathcal{P}}_a, \breve{\mathcal{P}}_b] = 0, \quad a, b = 1, 2, \tag{4.37}$$

$$\{\breve{\mathcal{Q}}_a, \breve{\mathcal{Q}}_b\} = 2\delta_{a,b}\prod_{j=1}^{k}(\breve{\mathcal{H}} - \breve{z}_j)^2\prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j)^2\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j)^2, \tag{4.38}$$

$$\{\breve{\mathcal{S}}_a, \breve{\mathcal{S}}_b\} = 2\delta_{a,b}\prod_{j=k+1}^{2g+1}(\breve{\mathcal{H}} - \breve{z}_j)\prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j)^2\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j)^2, \tag{4.39}$$

$$\begin{aligned}\{\breve{\mathcal{S}}_a, \breve{\mathcal{Q}}_b\} = 2\delta_{ab}G_{2g}(\breve{\mathcal{H}})\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j)\prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j) + \\ \epsilon_{ab}\left(\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j) + \prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j)\right)\breve{\mathcal{P}}_1 + \\ \epsilon_{ab}\left(\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j) - \prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j)\right)\breve{\mathcal{P}}_2,\end{aligned} \tag{4.40}$$

$$\begin{aligned}[\breve{\mathcal{P}}_a, \breve{\mathcal{Q}}_b] = (-1)^{g+1}i\left(\prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j) + (-1)^a\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j)\right)\times \\ \times\left(\breve{\mathcal{S}}_b\prod_{j=1}^{k}(\breve{\mathcal{H}} - \breve{z}_j) - G_{2g}(\breve{\mathcal{H}})\breve{\mathcal{Q}}_b\right),\end{aligned} \tag{4.41}$$

$$[\breve{\mathcal{P}}_a, \breve{\mathcal{S}}_b] = (-1)^{g+1}i\left(\prod_{j=1}^{l}(\breve{\mathcal{H}} - z_j) + (-1)^a\prod_{j=1}^{l'}(\breve{\mathcal{H}} - z'_j)\right)\times \tag{4.42}$$

$$\times\left(\breve{\mathcal{Q}}_b\prod_{j=k+1}^{2g+1}(\breve{\mathcal{H}} - \breve{z}_j) - G_{2g}(\breve{\mathcal{H}})\breve{\mathcal{S}}_b\right), \tag{4.43}$$

esta supersimetría está rota si alguna energía $\breve{z}_i$ es más baja que la menor entre $\{z_i\}$, $\{z'_i\}$ y $\{E_i\}$

Esté superálgebra es reducido de forma espontánea cuando se cumplen las siguientes coincidencias espectrales $\mathcal{D}_k\psi_{a_{j,1},a_{j,2}}(r_{j,1},r_{j,2},x) = \mathcal{D}_k\psi_{a'_{j',1},a'_{j',2}}(r'_{j',1},r'_{j',2},x)$ acá $j' = 1,2,\ldots,l'$ y $j = 1,2,\ldots,l$, o $\mathcal{B}_{2g+1-k}\psi_{a_{j,1},a_{j,2}}(r_{j,1},r_{j,2},x) = \mathcal{D}_k\psi_{a'_{j',1},a'_{j',2}}(r'_{j',1},r'_{j',2},x)$.

Sin perder generalidad estudiaremos el caso en que $\mathcal{D}_k\psi_{a_{j,1},a_{j,2}}(r_{j,1},r_{j,2},x) = \mathcal{D}_k\psi_{a'_{j,1},a'_{j,2}}(r'_{j,1},r'_{j,2},x)$ para $j = 1,2,\ldots,n$ y $\mathcal{B}_k\psi_{a_{j,1},a_{j,2}}(r_{j,1},r_{j,2},x) = \mathcal{D}_k\psi_{a'_{j,1},a'_{j,2}}(r'_{j,1},r'_{j,2},x)$ para $j = n+1,\ldots,n+m$. Usando la identidad (4.74) podemos demostrar que

$$\mathcal{D}_k\mathbb{A}^{\#}_n = \mathbb{A}'^{\mathcal{D}_k\#}_n\breve{\mathcal{D}}_{k;n}, \tag{4.44}$$

con $\breve{\mathcal{D}}_{k;n}$ un operador de entrelazamiento entre $H_{g,n}$ y $\breve{H}'_{g,n}$, $\breve{\mathcal{D}}_{k;n}H_{g,n} = \breve{H}'_{g,n}\breve{\mathcal{D}}_{k;n}$ y $H_{g,n}\breve{\mathcal{D}}^{\#}_{k;n} = \breve{\mathcal{D}}^{\#}_{k;n}\breve{H}'_{g,n}$, el cual reduce (4.32) en la forma

$$\mathcal{D}_{l',k,l} \to \mathcal{D}_{l',k,l;n}\prod_{j=1}^{n}(H_{g,l}-z_i)^2, \tag{4.45}$$

con

$$\mathcal{D}_{l',k,l;n} = \mathbb{A}'^{\mathcal{D}_k}_{n+1,l'}\breve{\mathcal{D}}_{k;n}\mathbb{A}^{\#}_{n+1,l}, \tag{4.46}$$

y

$$\breve{\mathcal{D}}_{k;n} = X_{k;n} \equiv X_{1;n}(k)\times\cdots\times X_{1;n}(1), \tag{4.47}$$

$$X_{1;n}(k') = (X_{k'-1;n}\mathbb{A}_n\psi(\breve{r}_{k'}))\frac{d}{dx}\frac{1}{(X_{k'-1;n}\mathbb{A}_n\psi(\breve{r}_{k'}))}, \quad X_{1;n}(1) = (\mathbb{A}_n\psi(\breve{r}_1))\frac{d}{dx}\frac{1}{\mathbb{A}_n\psi(\breve{r}_1))}, \tag{4.48}$$

por otro lado para mostrar como el operador $\mathcal{B}_{l',2g+1-k,l}$ es reducido debemos introducir el superíndice π en A^{π}_j el cual significa que el orden de los estados en la construcción del operador A_j fue cambiado en la forma

$$\psi_{a_{i,1},a_{i,2}}(r_{i,1},r_{i,2}) \to \psi_{a_{\pi(i),1},a_{\pi(i),2}}(r_{\pi(i),1},r_{\pi(i),2}), \tag{4.49}$$

con π la permutación

$$\pi = \begin{pmatrix} 1 & \ldots & m & m+1 & \ldots & m+n & n+m+1 & \ldots & l \\ n+1 & \ldots & n+m & 1 & \ldots & n & n+m+1 & \ldots & l \end{pmatrix}, \tag{4.50}$$

bajo esta permutación $A^{\pi}_{j>m+n} = A_{j>m+n}$. De esta forma podemos describir la reducción de orden en la forma

$$\mathcal{B}^{\pi}_{l',2g+1-k,l} \to \mathcal{B}^{\pi}_{l',2g+1-k,l;m}\prod_{j=n+1}^{n+m}(H_{g,l}-z_i)^2, \tag{4.51}$$

con

$$\mathcal{B}^{\pi}_{l',2g+1-k,l;m} = \mathbb{A}'^{\mathcal{D}_k}_{n+m+1,l'}\mathbb{A}'^{\pi,\mathcal{D}_k}_{m+1,n+m}\breve{\mathcal{B}}^{\pi}_{2g+1-k,m}\mathbb{A}^{\pi\#}_{m+1,n+m}\mathbb{A}^{\#}_{n+m+1,l}, \tag{4.52}$$

y

$$\breve{\mathcal{B}}^{\pi}_{2g+1-k,m} = X^{\pi\#}_{1,m}(k+1)\times\cdots\times X^{\pi\#}_{1,m}(2g+1), \tag{4.53}$$

un operador de entrelazamiento irreducible de orden $2g+1-k$ entre los Hamiltonianos $H^{\pi}_{g,m}$ y $\breve{H}'^{\pi}_{g,m}$, cumpliendo las relaciones de entrelazamiento $\breve{\mathcal{B}}_{2g+1-k,m}H^{\pi}_{g,m} = \breve{H}'^{\pi}_{g,m}\breve{\mathcal{B}}_{2g+1-k,m}$ y $H^{\pi}_{g,m}\breve{\mathcal{B}}^{\#}_{2g+1-k,m} = \breve{\mathcal{B}}^{\#}_{2g+1-k,m}\breve{H}'^{\pi}_{g,m}$,

$$X^{\pi}_{1,m}(k') = (X^{\pi}_{k'-1;n}\mathbb{A}^{\pi}_{m}\psi(\breve{r}_{k'}))\frac{d}{dx}\frac{1}{(X^{\pi}_{k'-1;n}\mathbb{A}^{\pi}_{m}\psi(\breve{r}_{k'}))}, \quad X^{\pi}_{1,m}(1) = (\mathbb{A}^{\pi}_{m}\psi(\breve{r}_{1}))\frac{d}{dx}\frac{1}{\mathbb{A}^{\pi}_{m}\psi(\breve{r}_{1}))}. \tag{4.54}$$

Usando (4.45) y (4.51) podemos escribir las siguientes integrales irreducibles de movimiento para $\breve{\mathcal{H}}$

$$\breve{\mathcal{Q}}_{1;n} = \begin{pmatrix} 0 & \mathcal{D}^{\#}_{l',k,l;n} \\ \mathcal{D}_{l',k,l;n} & 0 \end{pmatrix}, \quad \breve{\mathcal{S}}_{1;m} = \begin{pmatrix} 0 & \mathcal{B}^{\pi\#}_{l',2g+1-k,l;m} \\ \mathcal{B}^{\pi}_{l',2g+1-k,l;m} & 0 \end{pmatrix}, \tag{4.55}$$

Las integrales de Lax-Novikov no cambian su forma. El subsuperálgebra en este caso se ve reducido a la forma

$$[\breve{\mathcal{H}}, \breve{\mathcal{Q}}_{a;n}] = [\breve{\mathcal{H}}, \breve{\mathcal{S}}_{a,m}] = [\breve{\mathcal{H}}, \breve{\mathcal{P}}_{a}] = [\breve{\mathcal{H}}, \sigma_3] = [\breve{\mathcal{P}}_{a}, \breve{\mathcal{P}}_{b}] = 0, \quad a,b = 1,2, \tag{4.56}$$

$$\{\breve{\mathcal{Q}}_{a;n}, \breve{\mathcal{Q}}_{b;n}\} = 2\delta_{a,b}\prod_{j=1}^{k}(\breve{\mathcal{H}} - \breve{z}_j)^2 \prod_{j=n+1}^{l}(\breve{\mathcal{H}} - z_j)^2 \prod_{j=n+1}^{l'}(\breve{\mathcal{H}} - z'_j)^2, \tag{4.57}$$

$$\{\breve{\mathcal{S}}_{a,m}, \breve{\mathcal{S}}_{a,m}\} = 2\delta_{a,b}\prod_{j=k+1}^{2g+1}(\breve{\mathcal{H}} - \breve{z}_j) \prod_{j=m+1}^{l}(\breve{\mathcal{H}} - z_{\pi}(j))^2 \prod_{j=m+1}^{l'}(\breve{\mathcal{H}} - z'_{\pi}(j))^2, \tag{4.58}$$

$$\begin{aligned}\{\breve{\mathcal{S}}_{a;m}, \breve{\mathcal{Q}}_{b;n}\} = 2\delta_{ab}G_{2g;m;n}(\breve{\mathcal{H}}) \prod_{j=n+m+1}^{l'}(\breve{\mathcal{H}} - z'_j) \prod_{j=n+m+1}^{l}(\breve{\mathcal{H}} - z_j) + \\ \epsilon_{ab}\left(\prod_{j=n+m+1}^{l'}(\breve{\mathcal{H}} - z'_j) + \prod_{j=n+m+1}^{l}(\breve{\mathcal{H}} - z_j)\right)\breve{\mathcal{P}}_1 + \\ \epsilon_{ab}\left(\prod_{j=n+m+1}^{l'}(\breve{\mathcal{H}} - z'_j) - \prod_{j=n+m+1}^{l}(\breve{\mathcal{H}} - z_j)\right)\breve{\mathcal{P}}_2,\end{aligned} \tag{4.59}$$

$$[\breve{\mathcal{P}}_a, \breve{\mathcal{Q}}_{b;n}] = (-1)^{g+1}i\left(\prod_{j=n+m+1}^{l}(\breve{\mathcal{H}} - z_j) + (-1)^a \prod_{j=n+m+1}^{l'}(\breve{\mathcal{H}} - z'_j)\right) \times \tag{4.60}$$

$$\times\left(\breve{\mathcal{S}}_{b;m}\prod_{j=n+1}^{n+m}(\breve{\mathcal{H}} - z_j)^2 \prod_{j=1}^{k}(\breve{\mathcal{H}} - \breve{z}_j) - G_{2g;m;n}(\breve{\mathcal{H}})\breve{\mathcal{Q}}_{b;n}\right), \tag{4.61}$$

$$[\breve{\mathcal{P}}_a, \breve{\mathcal{S}}_{b;m}] = (-1)^{g+1}i\left(\prod_{j=m+1}^{l}(\breve{\mathcal{H}} - z_{\pi(j)}) + (-1)^a \prod_{j=m+1}^{l'}(\breve{\mathcal{H}} - z'_{\pi(j)})\right) \times \tag{4.62}$$

$$\times\left(\breve{\mathcal{Q}}_{b;n}\prod_{j=1}^{n}(\breve{\mathcal{H}} - z_j)^2 \prod_{j=k+1}^{2g+1}(\breve{\mathcal{H}} - \breve{z}_j) - G_{2g;m;n}(\breve{\mathcal{H}})\breve{\mathcal{S}}_{b;m}\right), \tag{4.63}$$

acá $G_{2g;m;n}$ es un polinomio definido en la forma

$$2G_{2g;m;n}(H_{g,n+m}) = \mathcal{D}^{\#}_{n+m,k,n+m;n}\mathcal{B}^{\pi}_{n+m,2g+1-k,n+m;m} + \tag{4.64}$$

$$\mathcal{B}^{\pi\#}_{n+m,2g+1-k,n+m;m}\mathcal{D}_{n+m,k,n+m;n}, \tag{4.65}$$

el cual es una integral de movimiento de orden $2g + 2m + 2n$ para $H_{g,n+m}$, como la única integral adicional de $H_{g,n+m}$, es $P_{2g+2n+2m+1}(u_{g,n+m}, \partial\sigma(H_{g,n+m}))$ y tiene orden $2g+2m+2n+1$, entonces $G_{2g;m;n}(H_{g,n+m})$ solo puede ser un polinomio en el propio Hamiltoniano $H_{g,n+m}$.

Podemos observar que es ua característica del cuadro supersimetrico en general que las integrales fermiónicas factorizan a las integrales bosónicas. Además en el caso isospectral $\sigma(H_{g,l}) = \sigma(H'_{g,l})$, $l = l'$, siempre una de las integrales bosónicas se convierte en un elemento central del superálgebra. Este resultado en particular juega un rol fundamental en la busqueda de soluciones para modelos de interacción no lineal entre fermiones, ya que conecta algunos cuadros superalgebraicos con soluciónes de la jerarquía de ecuaciones estacionarias e inhomogéneas de mKdV. En el trabajo adjunto al final de este capítulo se ha estudiado este problema en especifico para potenciales transparentes ($g = 0$) conectando este cuadro superalgebraico con condensados auto consistentes para los modelos de Bogoliubov-de Gennes y el de Gross-Neveu, diferenciando entre dos familias de condensados según su topología, dependiendo esta, de cual de los dos tipos de superálgebra mostradas en las subsecciónes anteriores rige la supersimetria exótica asociada a dicho condensado. También es estudiado el efecto de dispersión de solitones y como conecta estas dos familias para ciertos limites en los datos espectrales, lo cual pasamos a detallar a continuación.

4.3. Reducción espontánea de orden en operadores, transmutación de superálgebra de Lie y rompimiento espontáneo de simetrías

Las auto transformaciones de Darboux pueden ser interpretadas como un límite de una transformación solitónica (tomando el límite a_1 o a_2 igual a cero o ∞), ya que corresponde a la interacción no lineal de un solitón con energía $z(r)$ que desaparece en $x = \pm\infty$ según la carta a la que pertence r. En este caso, el defecto desaparece y el operador P_{2g+2+1} experimenta una reducción espontánea de orden en la forma $\hat{P}_{2g+2+1} \to X(r)P_{2g+1}X(r)^{\#} = (\tilde{H}^r - z(r))\tilde{P}^r_{2g+1}$. Estas reducciones de orden espontáneas, bajo cambios en los parámetros espectrales, enriquecen la estructura de las simetrías en que aparecen los potenciales transparentes ($g = 0$) o finite(g)-gap, ya que la misma simetría observa una dinámica en dependencia de los datos espectrales.

Por otro lado para soluciones que ya contienen solitones también son posibles transformaciones de Darboux de tipo anti-solitónicas: las cuales bajan en uno el orden de la ecución estacionaria de la jerarquía de KdV que satisface el potencial transformado con respecto al orden

de la ecuación de la jerarquía que soluciona el potencial inicial. Por ejemplo: usando el estado $\frac{1}{\psi_{a_1,a_2}(r_1,r_2,x)}$ de $\hat{H}$ con energía $z(r)$ podemos borrar el solitón en $\hat{u}$ reconstruyendo $u(x)$ nuevamente. En este caso el vestimiento de $\hat{P}^r_{2g+2+1}$ es una integral $I = A^{\#}\hat{P}^r_{2g+2+1}A$ formal de H pero si recordamos la factorización de H en función de los operadores de entrelazamiento más la correspondiente ecuación de la jerarquía de KdV en su formulación de par de Lax otenemos la siguiente reducción de orden para I

$$I = (H - z(r))^2 P_{2g+1} \to P_{2g+1}, \tag{4.66}$$

siendo en realidad P_{2g+1} la integral de Lax-Novikov irreducible para H, como era de esperarse.

La integral $P_{2g+2l+1}$ muestra una reducción de orden espontánea en dependencia del valor de los datos de disperción en la construcción de H_{g+l}. En los límites $a_{i,(1,2)}$ tiende a cero, o lo que es lo mismo $\psi_{a_{i,1},a_{i,2}}(r_{i,1}, r_{i,2}, x) \to \psi(r_{i,(2,1)})$ o en el límite $a_{i,b}$ tiende a infinito , o que es lo mismo $\psi_{a_{i,1},a_{i,2}}(r_{i,1}, r_{i,2}, x) \to \psi(r_{i,b})$ with $b = 1, 2$. la i-ésima transformación de Darboux solitónica es espontáneamente transformada en una auto transformación de Darboux permitiendo una reducción de orden espontánea del operador $P_{2g+2l+1}$. Podemos resumir este límite en la forma $\psi_{a_{i,1},a_{i,2}}(r_{i,1}, r_{i,2}, x) \to \psi(r_{i,b_i})$, con b_i igual a 1 o 2. En este límite los potenciales sufren la siguiente transformación

$$u_{g,j<i} \to \tilde{u}^{r_{i,b_i}}_{g,j} = \tilde{u}^{r_{i,b_i}}(x) - 2\frac{d^2}{dx^2}\ln(\ \tilde{\mathbb{W}}_j\ ), \tag{4.67}$$

$$u_{g,j\geq i} \to \tilde{u}^{r_{i,b_i}}_{g,j-1} = \tilde{u}^{r_{i,b_i}}(x) - 2\frac{d^2}{dx^2}\ln(\ \tilde{\mathbb{W}}_{j-1,i}\ ), \tag{4.68}$$

y los operadores de entrelazamiento cambian en la forma

$$A_{j<i} \to \tilde{A}^{r_{i,b_i}}_j = \frac{\tilde{\mathbb{W}}_j}{\tilde{\mathbb{W}}_{j-1}}\frac{d}{dx}\frac{\tilde{\mathbb{W}}_j}{\tilde{\mathbb{W}}_{j-1}}, \tag{4.69}$$

$$\tilde{\mathbb{W}}_j = \mathbb{W}(X(r_{i,b_i})\psi_{a_{1,1},a_{1,2}}(r_{1,1}, r_{1,2}, x), \dots X(r_{i,b_i})\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x)), \tag{4.70}$$

$$A_{j>i} \to \tilde{A}^{r_{i,b_i}}_{j-1} = \frac{\tilde{\mathbb{W}}_{j,i}}{\tilde{\mathbb{W}}_{j-1,i}}\frac{d}{dx}\frac{\tilde{\mathbb{W}}_{j,i}}{\tilde{\mathbb{W}}_{j-1,i}}, \tag{4.71}$$

$$\begin{aligned}\tilde{\mathbb{W}}_{j,i} = \quad & \mathbb{W}(X(r_{i,b_i})\psi_{a_{1,1},a_{1,2}}(r_{1,1}, r_{1,2}, x), \dots, \\ & X(r_{i,b_i})\psi_{a_{i-1,1},a_{i-1,2}}(r_{i-1,1}, r_{i-1,2}, x), \\ & X(r_{i,b_i})\psi_{a_{i+1,1},a_{i+1,2}}(r_{i+1,1}, r_{i+1,2}, x), \dots, \\ & X(r_{i,b_i})\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x)),\end{aligned} \tag{4.72}$$

acá $X(r_{i,b_i})\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x)$ son estados del Hamiltoniano $\tilde{H}^{r_{i,b_i}}_{g,0} = \tilde{H}^{r_{i,b_i}}$ con la misma energía que los estados $\psi_{a_{j,1},a_{j,2}}(r_{j,1}, r_{j,2}, x)$ de $H_{g,0}$ ($X(r_{i,b_i})H_{g,0} = \tilde{H}^{r_{i,b_i}}_{g,0}X(r_{i,b_i})$). El i-ésimo operador

de entrelazamiento se convierte en un operador de tipo desfase en la forma $\mathbb{A}_i \to X_{i-1}(r_{i,b_i})$, acá $X_{j<i}(r_{i,b_i})$ es un operador de Darboux de tipo dasfase que entrelaza $H_{g,j<i}$ y

$$\tilde{H}_{g,j<i}(r_{i,b_i}) = H(u(x) \to \tilde{u}^{r_{i,b_i}}_{g,j<i}), \tag{4.73}$$

y satisface

$$X_k(r_{i,b_i})A_k = \tilde{A}_k(r_{i,b_i})X_{k-1}(r_{i,b_i}), \quad X_0(r) = X(r), \tag{4.74}$$

y

$$X_k(r) = (\mathbb{A}_k\psi(r))\frac{d}{dx}\frac{1}{(\mathbb{A}_k\psi(r))}. \tag{4.75}$$

Usando esta relación entre operadores de Darboux de tipo solitónicos y de tipo auto Darboux o desfase, podemos mostrar fácilmente que en este límite la integral de Lax-Novikov de $H_{g,l}$ muestra una reduccón de orden espontánea en la forma

$$\begin{aligned} P_{2g+2l+1}(u_{g,l}, \partial\sigma H_{g,l}) \to \quad & (\tilde{H}_{g,l-1}(r_{i,b_i}) - z(r_{i,b_i}))^2 \times \\ & P_{2g+2(l-1)+1}(\tilde{u}^{r_{i,b_i}}_{g,l-1}, \partial\sigma H_{g,l}/\{z(r_{i,b_i}), z(r_{i,b_i})\}), \end{aligned} \tag{4.76}$$

acá $P_{2g+2(l-1)+1}(\tilde{H}^{r_{i,b_i}}_{g,l-1}, \partial\sigma H_{g,l}/\{z(r_{i,b_i}), z(r_{i,b_i})\})$ conmuta con $\tilde{H}^{r_{i,b_i}}_{g,l-1}$ y su relación de Burchnal-Chaundy nos da la superálgebra nolineal reducida (en dos ordenes)

$$[\tilde{H}, \tilde{\mathbb{P}}_a] = 0, \quad \{\tilde{\mathbb{P}}_a, \tilde{\mathbb{P}}_b\} = 2\delta_{ab}\prod_{j=0}^{2g}(\tilde{H} - E_j)\prod_{i\neq j=1}^{l}(\tilde{H} - z_j)^2, \tag{4.77}$$

acá $\tilde{H} = \tilde{H}_{g,l-1}(r_{i,b_i})$ y $\tilde{\mathbb{P}}_1 = P_{2g+2(l-1)+1}(\tilde{u}^{r_{i,b_i}}_{g,l-1}, \partial\sigma H_{g,l}/\{z(r_{i,b_i}), z(r_{i,b_i})\})$, $\Gamma\tilde{\mathbb{P}}_1$.

Otro ejemplo de reducción es cuando $z_{i+1} \to z_i$. En este límite un solitón elimina al otro. Existe un pseudo principio de exclusión entre solitones, ya que no pueden existir dos solitones con idéntica energía, tal como no puede haber degeneración dos en un energía ligada para un operador de Schrödinger no extendido, debido a que si un estado es normalizable su densidad de probabilidad tiende a cero cuando la coordenada espacial tiende a infinito. Formalmente la segunda solución de misma energía es inversamente proporcional a esta solución, por lo cual debe crecer exponencialmente cuando nos acercamos a los infinitos espaciales. En este límite el potencial observa los siguientes cambios

$$\begin{aligned} u_{g,l} \to \breve{u}_{g,l-2} = u_{g,0} - 2\frac{d^2}{dx^2}\ln(\quad & \mathbb{W}(\psi_{a_{1,1},a_{1,2}}(r_{1,1}, r_{1,2}, x), \ldots, \\ & \psi_{a_{i-1,1},a_{i-1,2}}(r_{i-1,1}, r_{i-1,2}, x), \\ & \psi_{a_{i+2,1},a_{i+2,2}}(r_{i+2,1}, r_{i+2,2}, x), \\ & \ldots, \psi_{a_{l,1},a_{l,2}}(r_{l,1}, r_{l,2}, x))), \end{aligned} \tag{4.78}$$

y el operador de entrelazamiento A_{i+1} cambia en la forma $A_{i+1} \to -A_i^{\#}$ permitiendo la factorización de Hamiltonianos en la estructura de P y por lo tanto una reducción de orden espontánea.

Para mantener las características físicas de u debemos demandar que no existe $\epsilon \in \partial\sigma(H_{g,l})$ tal que $z_i < \epsilon < z_{i+1}$ para $z_i < z_{i+1}$ o $z_i > \epsilon > z_{i+1}$ para $z_i > z_{i+1}$. En este caso la nolinealidad del superálgebra es reducida pero en orden cuatro.

$$[\breve{H}, \breve{\mathbb{P}}_a] = 0, \quad \{\breve{\mathbb{P}}_a, \breve{\mathbb{P}}_b\} = 2\delta_{ab} \prod_{j=0}^{2g} (\breve{H} - E_j) \prod_{i,i+1\neq j=1}^{l} (\breve{H} - z_j)^2. \tag{4.79}$$

En la siguiente publicación se muestra de forma muy ilustrativa como los potenciales y superpotenciales en fondo libre se deforman en los limites indicados anteriormente. Las dos familias más generales de potenciales transparentes escalares para operador de Dirac estacionario en 1+1D son estudiadas. Los efectos de dispersión son estudiados mostrando como se pasa de una familia de potenciales a la otra, a nivel de solitones de KdV estos limites corresponden a hacer desaparecer solitones en los infinitos espaciales. Ambas familias de potenciales de Dirac poseen una integral de movimiento correspondiente a la integral de Lax-Novikov asociada al par de Lax de la jerarquía de mKdV y permite reconstruir la supersimetría extendida $N = 4$ observada para pares de operadores de Schrödinger, ahora para pares de operadores de tipo Dirac.

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Transmutations of supersymmetry through soliton scattering and self-consistent condensates

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We consider the two most general families of the $(1+1)$D Dirac systems with transparent scalar potentials and two related families of the paired reflectionless Schrödinger operators. The ordinary $\mathcal{N}=2$ supersymmetry for such Schrödinger pairs is enlarged up to an exotic $\mathcal{N}=4$ nonlinear centrally extended supersymmetric structure, which involves two bosonic integrals composed from the Lax-Novikov operators for the stationary Korteweg–de Vries hierarchy. Each associated single Dirac system displays a proper $\mathcal{N}=2$ nonlinear supersymmetry with a nonstandard grading operator. One of the two families of the first- and second-order systems exhibits the unbroken supersymmetry, while another is described by the broken exotic supersymmetry. The two families are shown to be mutually transmuted by applying a certain limit procedure to the soliton scattering data. We relate the topologically trivial and nontrivial transparent potentials with self-consistent inhomogeneous condensates in the Bogoliubov–de Gennes and Gross-Neveu models and indicate the exotic $\mathcal{N}=4$ nonlinear supersymmetry of the paired reflectionless Dirac systems.

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I. INTRODUCTION

The Schrödinger and Dirac equations with reflectionless or soliton potentials are exactly solvable. The reflectionless potentials of a general form for the one-dimensional Schrödinger equation were obtained for the first time by Kay and Moses by solving the problem of a theoretical construction of a solid dielectric medium which is perfectly transparent to electromagnetic radiation [1]. Such perfectly transparent potentials appear in the $(1+1)$-dimensional Gross-Neveu (GN) model [2–4] and are closely related with a nonlinear problem of self-consistency of the Bogoliubov–de Gennes (BdG) equations [5–7]. They find applications in the description of a broad spectrum of phenomena in diverse areas of physics such as conducting polymers [8–13], charge fractionalization [14–17], and superconductivity [5–7], [18–22], just to mention a few. There is also a great interest in supersymmetry associated with fermions in soliton backgrounds [23–28].

Reflectionless potentials play a fundamental role in the theory of integrable systems. They appear as soliton solutions, particularly, to the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations. Their explicit form can be obtained by means of the inverse scattering method, by Bäcklund transformation, or by Darboux-Crum transformations [1], [29–41]. A characteristic feature of the two last methods is a possibility to construct these potentials from simple (formal) solutions of the free particle.

In the present work, we focus on the Darboux transformations. In this picture, there appear the first-order differential operators, which intertwine reflectionless Schrödinger and perfectly transparent Dirac Hamiltonians. This will allow us, following the line of Refs. [34–36,42–45], to study the interrelations between the exotic nonlinear supersymmetric structures emerging in the first- and second-order quantum reflectionless systems of the most general form corresponding to the KdV and mKdV solitons.[1] We also will observe an interesting phenomenon of transmutation of supersymmetry associated with the soliton scattering and will relate the construction to the self-consistent inhomogeneous condensates appearing in the GN and BdG models.

A relation of the soliton potentials with the GN model [2] goes back to the famous result of Dashen, Hasslacher, and Neveu [3], who found that minimizing the effective action of the model for the "condensate function" $\sigma(x)=-g\bar{\psi}\psi$ results in the condition that the Schrödinger potentials $U_{\pm}(x)$ given in terms of the Miura transformation [40], $U_{\pm}(x)\equiv g^2\sigma^2(x)\pm gd\sigma(x)/dx$, have to be reflectionless. On the other hand, the Dirac system with transparent potential $\sigma(x)$ appears in the Takayama–Lin-Liu–Maki (TLM) model for conducting polymers [8], which is a continuous model for solitons in polyacetylene, where the kink and kink-antikink solutions were found [12]. Though these two models have distinct physical interpretations,

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[1]For the earlier studies related to the appearance of the exotic extended supersymmetric structure in such a class of systems characterized by the presence of the nontrivial Lax-Novikov integral, see also Refs. [46–50].

they are equivalent mathematically, and the methods developed in the context of the GN model were applied in the study of the TLM model [9,11,13]. In general, the self-consistent solutions of the GN model are related with the Ablowitz-Kaup-Newell-Segur hierarchy [45,51] and by the same reason are intimately related with integrable systems in $1+1$ dimensions. Particularly, some solutions to the GN model were found to be related with the breather-type solutions of the sinh-Gordon and nonlinear Schrödinger equations [52].

The integrability of the equations of the KdV and mKdV hierarchies can be associated with the existence of an auxiliary spectral problem given in terms of the spectral operator H and the evolution generator P_j. The consistency condition appears there in the form of the equation for the Lax pair (H, P_j), $\frac{dH}{dt_j} = [H, P_j]$, which is equivalent to the corresponding evolution equation. For the jth equation of the KdV hierarchy, $H = -\frac{d^2}{dx^2} + U$ is the Schrödinger operator, while P_j, $j = 0, 1, \ldots$, is an anti-Hermitian monic differential operator of the form $P_j = \frac{d^{2j+1}}{dx^{2j+1}} + a_{2j-1}\frac{d^{2j-1}}{dx^{2j-1}} + \cdots + a_0$ with coefficient functions a_i given in terms of the potential U and its x derivatives. The case of the KdV equation corresponds to $j = 1$, and its n-soliton solution $U_n(x,t)$ satisfies simultaneously the equation $[\mathcal{L}_n, H_n] = 0$, which is the nonlinear ordinary differential equation of the order of $2n+1$ in the x variable. This is the nth stationary equation for the KdV hierarchy, in which $t_1 = t$ plays a role of an external parameter. The operator $\mathcal{L}_n = P_n + \sum_{j=0}^{n-1} c_j P_j$, where c_j are some real coefficients, is the Lax-Novikov nontrivial integral of motion for $H_n = -\frac{d^2}{dx^2} + U_n$ [32,33,48]. According to a celebrated result of Burchnall and Chaundy [41], the square of the order of the $2n+1$ differential operator $\mathcal{L}_n$ reduces to a certain polynomial in H_n.

One can construct the pair $(H_n, \mathcal{L}_n)$ corresponding to an n-soliton potential U_n recursively, starting from the free particle case with $H_0 = -\frac{d^2}{dx^2}$ and $\mathcal{L}_0 = \frac{d}{dx}$ $(U_0 = 0)$ and using the Darboux transformations. If we restrict ourselves by regular on the x-axis potentials, then at each step,

(i) from U_n, we construct an almost isospectral reflectionless potential U_{n+1} with one more bound state in comparison with U_n, and from the Lax-Novikov integral $\mathcal{L}_n$ for $H_n = -\frac{d^2}{dx^2} + U_n$, we obtain the integral $\mathcal{L}_{n+1}$ for reflectionless Schrödinger system H_{n+1}.

The interesting point here is that having reflectionless Schrödinger potential U_n of a general form, by applying the Darboux transformation of another nature,

(ii) we can construct from U_n another n-soliton potential $\tilde{U}_n$ to be completely isospectral to U_n, and from $\mathcal{L}_n$ we can obtain the corresponding integral $\tilde{\mathcal{L}}_n$ for $\tilde{H}_n$.

The latter construction can be realized by applying a certain limit procedure for soliton scattering data of the reflectionless potential U_{n+1}. By a similar limit procedure, one can also relate $\tilde{U}_n$ with U_{n-1}, and U_n with U_{n-2}. In both cases (i) and (ii) above, one can associate with the corresponding pairs of the reflectionless second-order Hamiltonians the exotic $\mathcal{N} = 4$ nonlinear supersymmetry that includes two bosonic integrals composed from Lax-Novikov integrals for the partner subsystems.

Exploiting the knowledge of the Darboux transformations for the KdV, one can generalize the construction for the case of the mKdV to get the transparent Dirac systems with the multikink scalar potentials and to identify for each such single first-order matrix system a proper exotic $\mathcal{N} = 2$ supersymmetry. As in the Schrödinger case, the transparent Dirac multikink potentials are separated into two groups: one of them is formed by topological and another by nontopological scalar potentials. The topological potentials are associated with case (i) above and represent the configurations of n kinks and $n \pm 1$ antikinks. The nontopological transparent potentials correspond to case (ii) and represent a certain superposition of n kinks and n antikinks. We shall show how the kinks and antikinks in transparent Dirac potentials gather together in such a way that their amplitudes and phases are fixed by supersymmetry of the paired reflectionless Schrödinger systems.

The paper is organized as follows. In Sec. II, we review shortly the recursive construction of the multisoliton Schrödinger potentials of the most general form in terms of the Darboux transformations and describe the spectra of the corresponding reflectionless Schrödinger operators. We identify there the independent differential operators of the orders of 1 and $2n$, which intertwine the neighbor in a recursive chain Schrödinger Hamiltonians H_n and H_{n-1}, and find the Lax-Novikov integral of differential order $2n+1$ for H_n. In Sec. III, we describe another unique family of the reflectionless pairs $(H_n, \tilde{H}_n)$ with completely isospectral partners, which are also intertwined by the Darboux generators to be differential operators of the same orders 1 and $2n$, and find a certain limit procedure, related to the soliton scattering, which mutually transmutes the two indicated families of the pairs of the transparent Schrödinger systems. In Sec. IV, we study the exotic nonlinear supersymmetries of the two families of the Schrödinger systems composed from the reflectionless isospectral $(H_n, \tilde{H}_n)$ and almost isospectral (H_n, H_{n-1}) pairs and observe the transmutations between these two families through the soliton scattering. In Sec. V, we study the transparent Dirac systems associated with the two families of the superextended reflectionless Schrödinger systems, where we show that each single transparent Dirac system possesses its own exotic nonlinear supersymmetry. Section VI is devoted to the discussion of the obtained results and outlook. There we relate the perfectly transparent scalar Dirac potentials with the self-consistent inhomogeneous condensates appearing in the BdG and

GN models and indicate the exotic $\mathcal{N}=4$ nonlinear supersymmetry of the paired reflectionless Dirac systems.

II. REFLECTIONLESS SCHRÖDINGER POTENTIALS AND DARBOUX-CRUM TRANSFORMATIONS

Let $H_n=-d^2/dx^2+U_n(x)$ be a reflectionless Schrödinger system with a potential of a general $2n$-parametric form $U_n(x)=U_n(x;\kappa_1,\tau_1,\ldots,\kappa_n,\tau_n)$ such that $U_n(x)\to 0$ for $x\to\pm\infty$. Parameters κ_j, $j=1,2,\ldots,n$, $0<\kappa_1<\cdots<\kappa_n$, correspond to the energy levels of the n bound states, $E_j=-\kappa_j^2$. They also define the transmission amplitude in the scattering sector with $E=k^2\geq 0$: $t(k)=\prod_{j=1}^n\frac{k+i\kappa_j}{k-i\kappa_j}$, and so $|t(k)|=1$ for any real value of the wave number k. The parameters τ_j are related to the norming constants of the bound state solutions [33,53], and their variation provides an isospectral deformation of the quantum system.

From the viewpoint of the inverse scattering method, function $U_n(x;\kappa_1,\tau_1,\ldots,\kappa_n,\tau_n)$ corresponds to the instant image of the n-soliton solution $U_n(x,t)$ to the KdV equation $u_t-6uu_x+u_{xxx}=0$. For large negative and positive values of time t, the $U_n(x,t)$ can be represented as a superposition of n one-soliton solutions of the amplitudes $2\kappa_j^2$ propagating to the right at speeds $v_j=4\kappa_j^2$:

$$U_n(x,t)=-\sum_{j=1}^{n}2\kappa_j^2\operatorname{sech}^2\kappa_j(x-4\kappa_j^2t-\varphi_j^{\pm})\quad\text{for}\quad t\to\pm\infty. \tag{2.1}$$

The phases $\varphi_j^{\pm}$ defined for $t\to\pm\infty$ are given by [33,53]

$$\varphi_l^{\pm}=\tau_l^0\pm\frac{1}{2\kappa_l}\left\{\sum_{j=l+1}^{n}\log\left|\frac{\kappa_l+\kappa_j}{\kappa_l-\kappa_j}\right|-\sum_{j=1}^{l-1}\log\left|\frac{\kappa_l+\kappa_j}{\kappa_l-\kappa_j}\right|\right\}, \tag{2.2}$$

where it is implied that for $l=n$ and $l=1$ the first and, respectively, the second sum disappears. The parameter τ_l^0 corresponds to the mean of the asymptotic phases, $\tau_l^0=\frac{1}{2}(\varphi_l^{+}+\varphi_l^{-})$. According to (2.2), the solitons demonstrate in some sense a fermionlike behavior: $|\varphi_l^{\pm}|,|\varphi_{l+1}^{\pm}|\to\infty$ as soon as $\kappa_l\to\kappa_{l+1}$. In the two-soliton case, (2.2) gives

$$\varphi_1^{\pm}=\tau_1^0\pm\frac{1}{2\kappa_1}\log\left|\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\right|,\qquad \varphi_2^{\pm}=\tau_2^0\mp\frac{1}{2\kappa_2}\log\left|\frac{\kappa_1+\kappa_2}{\kappa_1-\kappa_2}\right|. \tag{2.3}$$

Our consideration will be based on the method of iterated Darboux transformations (or, that is the same, the Darboux-Crum transformations) [29], by which the quantum mechanical reflectionless system with n bound states can be constructed from a free particle system $H_0=-\frac{d^2}{dx^2}$:

$$H_n=H_0+U_n(x),\qquad U_n=-2\frac{d^2}{dx^2}\log\mathbb{W}_n. \tag{2.4}$$

Here $\mathbb{W}_n$ is the Wronskian of n formal (nonphysical) eigenstates ψ_j of H_0, $H_0\psi_j=-\kappa_j^2\psi_j$:

$$\mathbb{W}_n=\mathbb{W}(\psi_1,\ldots,\psi_n)=\det\mathcal{A},\qquad \mathcal{A}_{ij}=\frac{d^{i-1}}{dx^{i-1}}\psi_j,\qquad i,j=1,\ldots,n, \tag{2.5}$$

which are chosen as follows:

$$\psi_j=\begin{cases}\cosh(\kappa_j(x+\tau_j)) & \text{for } j=\text{odd},\\ \sinh(\kappa_j(x+\tau_j)) & \text{for } j=\text{even},\end{cases}\qquad 0<\kappa_1<\kappa_2<\cdots<\kappa_{j-1}<\kappa_n. \tag{2.6}$$

Eigenfunctions $\Psi_0(x;E)\neq\psi_j$ of H_0, $H_0\Psi_0(x;E)=E\Psi_0(x;E)$, are mapped into the eigenfunctions $\Psi_n(x;E)$ of H_n, $H_n\Psi_n(x;E)=E\Psi_n(x;E)$, by means of the Wronskian fractions:

$$\Psi_n(x;E)=\mathbb{W}(\psi_1,\ldots,\psi_n,\Psi_0(E))/\mathbb{W}_n. \tag{2.7}$$

The eigenfunctions in the scattering sector with $E=k^2\geq 0$, $k\geq 0$, and (not normalized) bound states with energies $E_j=-\kappa_j^2$, $j=1,\ldots,n$, of the system H_n are given then by the relations

$$\Psi_n^{\pm}(k^2)=\mathbb{W}(\psi_1,\ldots,\psi_n,e^{\pm ikx})/\mathbb{W}_n,\qquad \Psi_n(-\kappa_j^2)=\mathbb{W}\left(\psi_1,\ldots,\psi_n,\frac{d\psi_j}{dx}\right)/\mathbb{W}_n. \tag{2.8}$$

The derivative $\frac{d\psi_j}{dx}$ is a nonphysical eigenfunction of H_0 which is linearly independent from the corresponding nonphysical eigenfunction ψ_j from (2.6).

Coherently with (2.4), we put $\mathbb{W}_0=1$ and define the *prepotentials* Ω_n, $n=0,1,\ldots$:

$$\Omega_n = -\frac{d}{dx}\log \mathbb{W}_n \Rightarrow \frac{d}{dx}\Omega_n = \frac{1}{2}U_n. \tag{2.9}$$

Then we introduce the first-order differential operators

$$A_n = \frac{d}{dx} + \mathcal{W}_n, \qquad \mathcal{W}_n = \Omega_n - \Omega_{n-1}, \tag{2.10}$$

where, particularly, $\mathcal{W}_1 = \Omega_1 = -\kappa_1 \tanh \kappa_1(x+\tau_1)$. These operators and their conjugate ones factorize the reflectionless systems H_{n-1} and H_n having the $(n-1)$- and n-soliton potentials $U_{n-1}(x;\kappa_1,\tau_1,\ldots,\kappa_{n-1},\tau_{n-1})$ and $U_n(x;\kappa_1,\tau_1,\ldots,\kappa_{n-1},\tau_{n-1},\kappa_n,\tau_n)$:

$$A_n^\dagger A_n = H_{n-1} + \kappa_n^2, \qquad A_n A_n^\dagger = H_n + \kappa_n^2, \tag{2.11}$$

and intertwine them:

$$A_n H_{n-1} = H_n A_n, \qquad A_n^\dagger H_n = H_{n-1} A_n^\dagger. \tag{2.12}$$

The operator A_n can be presented equivalently as $A_n = \Psi^A_{n-1}\frac{d}{dx}(1/\Psi^A_{n-1})$, where $\Psi^A_{n-1} \equiv \frac{\mathbb{W}_n}{\mathbb{W}_{n-1}}$ is a nodeless nonphysical eigenfunction of H_{n-1}, $H_{n-1}\Psi^A_{n-1} = -\kappa_n^2\Psi^A_{n-1}$. This function is a formal (exponentially blowing up at $x = \pm\infty$) zero mode of the first-order differential operator A_n, $A_n\Psi^A_{n-1} = 0$. Any other (physical or nonphysical) eigenstate $\Psi_{n-1}(E)$ of H_{n-1}, $H_{n-1}\Psi_{n-1}(E) = E\Psi_{n-1}(E)$, is mapped by A_n into the eigenstate of H_n:

$$\Psi_n(E) = A_n\Psi_{n-1}(E), \tag{2.13}$$

with the same eigenvalue, $H_n\Psi_n(E) = E\Psi_n(E)$.

By iteration of (2.12), reflectionless system H_n can be related with the free particle H_0:

$$\mathbb{A}_n H_0 = H_n \mathbb{A}_n, \qquad \mathbb{A}_n^\dagger H_n = H_0 \mathbb{A}_n^\dagger, \tag{2.14}$$

where $\mathbb{A}_n$ is the differential operator of the order of n:

$$\mathbb{A}_n \equiv A_n \ldots A_1. \tag{2.15}$$

In terms of (2.15), we define the differential operator of the order of $2n$:

$$\mathcal{B}_1 = A_1\left(-\frac{d}{dx} + \kappa_1\right),$$
$$\mathcal{B}_n = \mathbb{A}_n\left(-\frac{d}{dx} + \kappa_n\right)\mathbb{A}_{n-1}^\dagger \quad \text{for} \quad n = 2, \ldots. \tag{2.16}$$

The iteration of relations (2.12) shows that $\mathcal{B}_n$ and $\mathcal{B}_n^\dagger$ also intertwine reflectionless Hamiltonians H_n and H_{n-1}:

$$\mathcal{B}_n H_{n-1} = H_n \mathcal{B}_n, \qquad \mathcal{B}_n^\dagger H_n = H_{n-1}\mathcal{B}_n^\dagger. \tag{2.17}$$

Unlike A_n and $A_n^\dagger$, they do this not directly but via the "virtual" free particle system H_0, for which the first-order differential operator $\frac{d}{dx}$ appearing explicitly in the structure of $\mathcal{B}_n$ is an integral of motion. Instead of (2.11), we have the relations

$$\mathcal{B}_n\mathcal{B}_n^\dagger = \prod_{j=1}^n (H_n + \kappa_j^2)^2, \qquad \mathcal{B}_n^\dagger\mathcal{B}_n = \prod_{j=1}^n (H_{n-1} + \kappa_j^2)^2. \tag{2.18}$$

The operator (2.15) also allows us to find a nontrivial integral for reflectionless system H_n:

$$\mathcal{L}_n = \mathbb{A}_n p \mathbb{A}_n^\dagger, \qquad \mathcal{L}_n^\dagger = \mathcal{L}_n, \qquad [\mathcal{L}_n, H_n] = 0. \tag{2.19}$$

This differential operator of the order of $2n+1$ is the Lax-Novikov integral for the H_n. It is a Darboux-dressed form of the integral $p = -i\frac{d}{dx}$ for the free particle system H_0, which satisfies the nonlinear supersymmetry-type relation

$$\mathcal{L}_n^2 = H_n \prod_{i=1}^n (H_n + \kappa_i^2)^2. \tag{2.20}$$

The property of commutativity of $\mathcal{L}_n$ with H_n means that the potential $U_n = 2\frac{d}{dx}\Omega_n$ is a solution of the nth member of the KdV stationary hierarchy.[2]

Using analogs of the integrals (2.19) for H_l with $0 < l < n$, one could try to construct the operators intertwining H_{n-1} and H_n with $n > 1$ via a virtual H_l system. In such a way we obtain, however, a combination of $\mathcal{B}_n$ and A_n with a coefficient before the latter operator to be a polynomial of the order of $(n-1)$ in H_{n-1}. For instance, $-iA_n\mathcal{L}_{n-1}$ is the differential operator of the order of $2n$, which, like $\mathcal{B}_n$, intertwines H_{n-1} with H_n but reduces to $-iA_n\mathcal{L}_{n-1} = \mathcal{B}_n - \kappa_n A_n\prod_{i=1}^{n-1}(H_{n-1} + \kappa_i^2)$ and, so, is not a new, independent intertwining operator. At the same time, note that the intertwining operators A_n and $\mathcal{B}_n$, and the integral $\mathcal{L}_n$ are related with the Hamiltonian H_n by the identity

$$\mathcal{B}_n A_n^\dagger + i\mathcal{L}_n = \kappa_n \prod_{i=1}^n (H_n + \kappa_i^2). \tag{2.21}$$

In conclusion of this section, it is worth stressing once more that the existence of the nontrivial, order $2n$ intertwining operator $\mathcal{B}_n$ in addition to the first-order Darboux generator A_n as well as of the order $2n+1$ integral $\mathcal{L}_n$ originates from the fact that the reflectionless system H_n is related to the free particle H_0 by the chain of the subsequent Darboux transformations, and the H_0 possesses a nontrivial integral of motion $p = -i\frac{d}{dx}$.

[2]Note that, unlike Sec. 1, we take $\mathcal{L}_n$ here in a Hermitian form.

III. SOLITON SCATTERING AND DARBOUX TRANSFORMATIONS

Besides the discussed pairs (H_n, H_{n-1}) of reflectionless Schrödinger systems related by the first-order Darboux intertwining operators, there is another class of such systems, for which the paired Hamiltonians are also interrelated by the first-order Darboux generators. Unlike the described case, the reflectionless partners in these pairs are completely isospectral. The corresponding n-soliton partner potentials $U_n(x;\kappa_1,\tau_1,\ldots,\kappa_n,\tau_n)$ and $U_n(x;\kappa_1,\tilde{\tau}_1,\ldots,\kappa_n,\tilde{\tau}_n)$ are characterized by the same scaling parameters κ_i, $i=1,\ldots,n$, but different sets of the translation parameters correlated as follows [35]:

$$\tau_i-\tilde{\tau}_i=\frac{1}{\kappa_i}\operatorname{arctanh}\frac{\kappa_i}{\mathcal{C}}=\frac{1}{2\kappa_i}\log\frac{\mathcal{C}+\kappa_i}{\mathcal{C}-\kappa_i}, \tag{3.1}$$

where $\mathcal{C}$ is an additional real parameter such that $|\mathcal{C}|>\kappa_n$. A comparison of the quantities (3.1) and (2.3) indicates that (3.1) can be related somehow to the effect of the scattering of solitons. In this section, we show how each indicated family of the paired reflectionless systems, with partners intertwined by the first-order Darboux generators, can be transformed into another by a certain limit procedure, which admits a soliton scattering interpretation.

To this aim, we first consider the limits $\tau_n\to\pm\infty$ applied to the reflectionless system H_n. To study the induced deformation of the potential U_n and superpotential $\mathcal{W}_n$ (the latter will play a role of the potential for an associated Dirac system), it is sufficient to investigate the limits of the prepotential Ω_n because of the relations $2\frac{d}{dx}\Omega_n=U_n$ and $\Omega_n-\Omega_{n-1}=\mathcal{W}_n$. We shall demonstrate that $\Omega_n=-\frac{d}{dx}\log\mathbb{W}_n\to\tilde{\Omega}_{n-1}(\mathcal{C})-\mathcal{C}$ for $\tau_n\to\pm\infty$, where $\mathcal{C}=\pm\kappa_n$ and $\tilde{\Omega}_{n-1}$ is identical to Ω_{n-1} with τ_i, $i=1,\ldots,n-1$, changed for $\tilde{\tau}_i=\tau_i-\frac{1}{2\kappa_i}\log\frac{\mathcal{C}+\kappa_i}{\mathcal{C}-\kappa_i}$. From here, it follows also that if we apply subsequently another limit $\kappa_n\to\kappa_{n-1}$, or that is the same, $\tilde{\tau}_{n-1}\to\mp\infty$, the deformed (by κ-dependent τ displacements) prepotential transforms as $(\tilde{\Omega}_{n-1}-\mathcal{C})\to\Omega_{n-2}$. So, the effect of sending subsequently the two solitons with indices n and $n-1$ to infinity in opposite directions results in the disappearance of the two bound states from the spectrum, without changing the rest of the $2(n-2)$ soliton parameters in the reflectionless potential U_{n-2}. This corresponds to a fermionlike behavior of solitons already mentioned below Eq. (2.2).

In the limit $\tau_n\to\pm\infty$, for the prepotential $\Omega_n=-(\log\mathbb{W}(\psi_1,\ldots,\psi_n))_x$ we find that $\Omega_n\to-(\log\mathbb{W}(\psi_1,\ldots,\psi_{n-1},C_n^{\pm}e^{\pm\kappa_n x}))_x$, where $C_n^{\pm}=\epsilon_n^{\pm}\frac{1}{2}e^{\pm\kappa_n\tau_n}$ is an exponentially divergent multiplicative factor with $\epsilon_n^{+}=1$ and $\epsilon_n^{-}=(-1)^{n+1}$. By the Wronskian properties, we have $\mathbb{W}(\psi_1,\ldots,\psi_{n-1},C_n^{\pm}e^{\pm\kappa_n x})=C_n^{\pm}\mathbb{W}(\psi_1,\ldots,\psi_{n-1},e^{\pm\kappa_n x})$. The logarithmic derivative eliminates the x-independent divergent multiplicative factor $C_n^{\pm}$, and in the limit $\tau_n\to\pm\infty$ we obtain $\Omega_n\to-(\log\mathbb{W}(\psi_1,\ldots,\psi_{n-1},e^{\pm\kappa_n x}))_x$. We note now that $\mathbb{W}(\psi_1,\ldots\psi_{n-1},e^{\pm\kappa_n x})=e^{\pm\kappa_n x}\det\|W_n^{\natural}\|$, where

$$\|W_n^{\natural}\|=\begin{pmatrix}\mathrm{ch}\kappa_1x_1 & \mathrm{sh}\kappa_2x_2 & \ldots & \psi_{n-1} & 1\\ \kappa_1\mathrm{sh}\kappa_1x_1 & \kappa_2\mathrm{ch}\kappa_2x_2 & & \partial_x\psi_{n-1} & \pm\kappa_n\\ \vdots & & \ddots & \vdots & \vdots\\ \partial_x^{n-1}\mathrm{ch}\kappa_1x_1 & \partial_x^{n-1}\mathrm{sh}\kappa_2x_2 & \ldots & \partial_x^{n-1}\psi_{n-1} & (\pm1)^{n-1}\kappa_n^{n-1}\end{pmatrix} \tag{3.2}$$

and $x_i\equiv x+\tau_i$. By changing the rows L_j, $j=1,\ldots,n-1$, of the matrix (3.2) for the linear combinations: $L_j\to\kappa_nL_j\mp L_{j+1}$, we find that $(\log\det\|W_n^{\natural}\|)_x=(\log\det\|\hat{W}_n\|)_x$ where

$$\|\hat{W}_n\|=\begin{pmatrix}\mathrm{Ch}_1^{\mp} & \mathrm{Sh}_2^{\mp} & \mathrm{Ch}_3^{\mp} & \ldots & 0\\ \kappa_1\mathrm{Sh}_1^{\mp} & \kappa_2\mathrm{Ch}_2^{\mp} & \kappa_3\mathrm{Sh}_3^{\mp} & \ldots & 0\\ \kappa_1^2\mathrm{Ch}_1^{\mp} & \kappa_2^2\mathrm{Sh}_2^{\mp} & \kappa_3^2\mathrm{Ch}_3^{\mp} & \ldots & 0\\ . & . & \ldots & . & \\ \partial_x^{n-1}\cosh\kappa_2x_2 & \partial_x^{n-1}\sinh\kappa_2x_2 & \partial_x^{n-1}\cosh\kappa_3x_3 & \ldots & (\pm1)^{n-1}\kappa_n^{n-1}\end{pmatrix}. \tag{3.3}$$

Here we denote $\mathrm{Ch}_i^{\mp}=\kappa_n\cosh\kappa_ix_i\mp\kappa_i\sinh\kappa_ix_i$ and $\mathrm{Sh}_i^{\mp}=\kappa_n\sinh\kappa_ix_i\mp\kappa_i\cosh\kappa_ix_i$, $i=1,\ldots,n-1$, where the signs $-$ and $+$ correspond to the limits $\tau_n\to+\infty$ and $\tau_n\to-\infty$, respectively. These functions can be represented equivalently as $\mathrm{Ch}_i^{\mp}=\sqrt{\kappa_n^2-\kappa_i^2}\cosh\kappa_i(x+\tau_i\mp\varphi_i)$ and $\mathrm{Sh}_i^{\mp}=\sqrt{\kappa_n^2-\kappa_i^2}\sinh\kappa_i(x+\tau_i\mp\varphi_i)$, where $\varphi_i=\frac{1}{2\kappa_i}\log\frac{\kappa_n+\kappa_i}{\kappa_n-\kappa_i}$, $i=1,\ldots,n-1$. As a consequence, we find that $(\log\mathbb{W}(\psi_1,\ldots,\psi_{n-1},e^{\pm\kappa_n x}))_x=\pm\kappa_n+(\log\mathbb{W}(\tilde{\psi}_1,\ldots,\tilde{\psi}_{n-1}))_x$, where $\tilde{\psi}_i$ is identical to ψ_i but with τ_i, $i=1,\ldots,n-1$, changed for

$$\tilde{\tau}_i = \tau_i \mp \frac{1}{2\kappa_i}\log\frac{\kappa_n+\kappa_i}{\kappa_n-\kappa_i} \quad \text{for} \quad \tau_n \to \pm\infty, \tag{3.4}$$

that translates finally into the transformation $\Omega_n \underset{\tau_n\to\pm\infty}{\to} \tilde{\Omega}_{n-1} \mp \kappa_n$. Note that $\tilde{\tau}_i - \tau_i$ given by (3.4) corresponds to the change of the phase in the two-soliton scattering given by the first relation in (2.3), with indices 1 and 2 changed for i and n, respectively.

In the limit $\tau_n \to +\infty$ we find that

$$A_n = \frac{d}{dx} + \mathcal{W}_n \to \frac{d}{dx} - \Delta_{n-1}(\kappa_n) = -X^\dagger_{n-1}(\kappa_n), \tag{3.5}$$

where

$$X_{n-1} = \frac{d}{dx} + \Delta_{n-1},$$
$$\Delta_{n-1}(\kappa_n) = \Omega_{n-1} - \tilde{\Omega}_{n-1}(\kappa_n) + \kappa_n. \tag{3.6}$$

The subsequent application of the limit $\kappa_n \to \kappa_{n-1}$ gives

$$X_{n-1}(\kappa_n) \to A_{n-1}, \qquad \tilde{A}_{n-1} \to X_{n-2}(\kappa_{n-1}), \tag{3.7}$$

where the first-order operator $\tilde{A}_{n-1}$ is of the same form as A_{n-1} but with the parameters τ_i changed for $\tilde{\tau}_i = \tau_i - \frac{1}{2\kappa_i}\log\frac{\mathcal{C}+\kappa_i}{\mathcal{C}-\kappa_i}$. The relations corresponding to the limit $\tau_n \to -\infty$ can be written down explicitly in a similar way.

Since the n-soliton potentials are given by the relation $U_n = 2\frac{d}{dx}\Omega_n$, by taking the limit $\tau_n \to +\infty$ we eliminate the bound state with $E_n = -\kappa_n^2$ from the spectrum of H_n and obtain a new Hamiltonian with $(n-1)$ bound states, which we call $\tilde{H}_{n-1}$. This Hamiltonian is isospectral to H_{n-1}, but each soliton in it is displaced with a phase dependent on κ_n:

$$H_n(\tau_i) \underset{\tau_n\to+\infty}{\to} H_{n-1}(\tilde{\tau}_i) \equiv \tilde{H}_{n-1}(\kappa_n),$$
$$\tilde{\tau}_i = \tau_i - \frac{1}{2\kappa_i}\log\frac{\kappa_n+\kappa_i}{\kappa_n-\kappa_i}. \tag{3.8}$$

The limit $\tau_n \to -\infty$ corresponds to the change of κ_n for $-\kappa_n$ in (3.8). In general, from the viewpoint of $\tilde{H}_{n-1}$, the κ_n (or $-\kappa_n$) is just an additional parameter, and from now on we call $\tilde{H}_{n-1} \equiv \tilde{H}_{n-1}(\mathcal{C})$, assuming for the sake of definiteness that $\mathcal{C} > \kappa_{n-1}$.

On the other hand, both the Hamiltonians H_n in the limit $\kappa_n \to \kappa_{n-1}$ and $\tilde{H}_{n-1}$ in the limit $\mathcal{C} \to \kappa_{n-1}$ correspond to a Hamiltonian H_{n-2}:

$$H_n \underset{\kappa_n\to\kappa_{n-1}}{\to} H_{n-2}, \qquad \tilde{H}_{n-1}(\mathcal{C}) \underset{\mathcal{C}\to\kappa_{n-1}}{\to} H_{n-2}. \tag{3.9}$$

As analogs of factorizations (2.11), we obtain

$$X_n^\dagger X_n = \tilde{H}_n + \mathcal{C}^2, \qquad X_n X_n^\dagger = H_n + \mathcal{C}^2, \tag{3.10}$$

where X_n is defined in (3.6) with index $n-1$ changed for n, and it is assumed here that $\mathcal{C}^2 > \kappa_n^2$. In correspondence with (3.10), X_n and $X_n^\dagger$ not only factorize the isospectral Hamiltonians, but also intertwine them: $X_n\tilde{H}_n = H_n X_n$, $X_n^\dagger H_n = \tilde{H}_n X_n^\dagger$. We also have the factorization relations

$$\tilde{A}_n\tilde{A}_n^\dagger = \tilde{H}_n + \kappa_n^2, \qquad \tilde{A}_n^\dagger\tilde{A}_n = \tilde{H}_{n-1} + \kappa_n^2. \tag{3.11}$$

Using these last relations, one can construct the generators which intertwine $\tilde{H}_n$ and H_n, being the differential operators of the order of $2n$:

$$\mathcal{Y}_n = \mathbb{A}_n\tilde{\mathbb{A}}_n^\dagger, \qquad \mathcal{Y}_n^\dagger = \tilde{\mathbb{A}}_n\mathbb{A}_n^\dagger, \tag{3.12}$$

$\mathcal{Y}_n\tilde{H}_n = H_n\mathcal{Y}_n$, $\mathcal{Y}_n^\dagger H_n = \tilde{H}_n\mathcal{Y}_n^\dagger$, where $\tilde{\mathbb{A}}_n$ is defined as in (2.15) but with A_i changed for $\tilde{A}_i$.

Another pair of important identities is

$$A_n X_{n-1} = X_n\tilde{A}_n, \qquad X_n^\dagger A_n = \tilde{A}_n X_{n-1}^\dagger. \tag{3.13}$$

The operators appearing in the first equality intertwine the Hamiltonians $\tilde{H}_{n-1}$ and H_n, $(A_nX_{n-1})\tilde{H}_{n-1} = H_n(A_nX_{n-1})$, $(X_n\tilde{A}_n)\tilde{H}_{n-1} = H_n(X_n\tilde{A}_n)$, and the equal operators from the other relation intertwine in a similar manner H_{n-1} and $\tilde{H}_n$. The Hermitian conjugate forms of the operators from (3.13) intertwine the indicated pairs of the Hamiltonians in the opposite direction. The relations in (3.13) are equivalent to the identity

$$(\mathcal{C} + \Omega_{n-1} - \tilde{\Omega}_n)(\Omega_n - \tilde{\Omega}_n - \Omega_{n-1} + \tilde{\Omega}_{n-1}) = (\tilde{\Omega}_n - \Omega_{n-1})_x, \tag{3.14}$$

which, in turn, is reduced to trigonometric identities [35]. In the limit $\tau_n \to \infty$, we find then that the intertwining between H_{n-1} and H_n operator $\mathcal{B}_n$ [see Eq. (2.17)] transforms into

$$\mathcal{B}_n \to (\tilde{H}_{n-1}(\kappa_n) + \kappa_n^2)\mathcal{Y}_{n-1}^\dagger(\kappa_n)$$
$$- 2\kappa_n\left(\prod_{i=1}^{n-1}(\tilde{H}_{n-1}(\kappa_n) + \kappa_i^2)\right)X_{n-1}^\dagger(\kappa_n). \tag{3.15}$$

This is a reducible intertwining operator for a pair H_{n-1} and $\tilde{H}_{n-1}$. From (3.15) we extract the irreducible operators $\mathcal{Y}_{n-1}^\dagger$ and $X_{n-1}^\dagger(\kappa_n)$ which intertwine the Hamiltonians H_{n-1} and $\tilde{H}_{n-1}$. At the same time, for the Lax-Novikov integral $\mathcal{L}_n$ we have

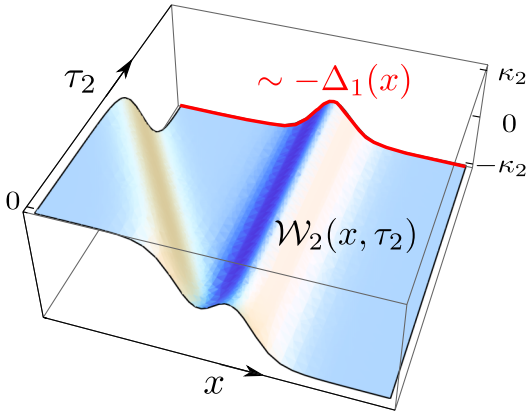

FIG. 1 (color online). In the limit $\tau_n \to \infty$, a topologically nontrivial superpotential $\mathcal{W}_n$ (being also the corresponding scalar Dirac potential) with asymptotic behavior $\lim_{x\to-\infty}\mathcal{W}_n(x) = -\lim_{x\to+\infty}\mathcal{W}_n(x) = \kappa_n > 0$ transforms (asymptotically) into a topologically trivial superpotential $-\Delta_{n-1}$ such that $\lim_{x\to-\infty}\Delta_{n-1}(x) = \lim_{x\to+\infty}\Delta_{n-1}(x) = \kappa_n > 0$. This corresponds to sending the nth kink to $x = -\infty$. The figure corresponds to the case $n = 2$ and shows the superpotential $\mathcal{W}_2$ as a function of x and τ_2.

$$\mathcal{L}_n \to (\tilde{H}_{n-1}(\kappa_n) + \kappa_n^2)\tilde{\mathcal{L}}_{n-1}, \tag{3.16}$$

that provides us with the irreducible nontrivial integral $\tilde{\mathcal{L}}_{n-1}$ for $\tilde{H}_{n-1}$.[3]

Figures 1 and 2 illustrate different limits for superpotentials $\mathcal{W}_n$ and Δ_n, while Figs. 3 and 4 show the transformations between potentials U_n and $\tilde{U}_n$.

We have considered the limit when the translation parameter τ_n in the n-soliton potential U_n tends to infinity. It is interesting to see what happens with reflectionless system H_n when we take the limit $\tau_j \to \pm\infty$ with $j < n$. Considering the same procedure as in the case $j = n$, we find that the prepotential Ω_n changes for Ω'_{n-1}, in which instead of (3.4) the arguments τ_i are replaced by

$$\tau_i{}' = \begin{cases} \tau_i \mp \frac{1}{2\kappa_i}\log\frac{\kappa_j+\kappa_i}{\kappa_j-\kappa_i} & \text{for } i < j, \\ \tau_i \pm \frac{1}{2\kappa_i}\left(\log\frac{\kappa_j+\kappa_i}{\kappa_i-\kappa_j} + i\pi\right) & \text{for } i > j. \end{cases} \tag{3.17}$$

For $i > j$ we have $\cosh\kappa_i(x+\tau_i') = \pm i\sinh\kappa_i(x+\hat{\tau}_i)$, $\sinh\kappa_i(x+\tau_i') = \pm i\cosh\kappa_i(x+\hat{\tau}_i)$, where

$$\hat{\tau}_i = \tau_i \mp \frac{1}{2\kappa_i}\log\left|\frac{\kappa_j+\kappa_i}{\kappa_j-\kappa_i}\right|. \tag{3.18}$$

[3]The questions of redundancy of nonlinear supersymmetric algebra in a general context were studied in [50]; see also the recent review [54]. The very nontrivial picture of redundancy and transmutations appearing in the completely isospectral supersymmetric pairs of reflectionless systems was investigated in detail in [34,35].

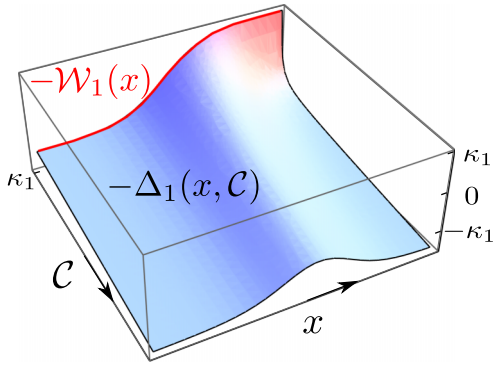

FIG. 2 (color online). A topologically trivial superpotential Δ_n transforms into a topologically nontrivial superpotential $\mathcal{W}_n$ through the limit $|\tilde{\tau}_n| \to \infty$, which is equivalent to the limit $\mathcal{C}^2 \to \kappa_n^2$. The figure illustrates the case when the kink-antikink Dirac potential with $n = 1$ transforms in the limit $\mathcal{C} \to \kappa_1$ into the antikink potential.

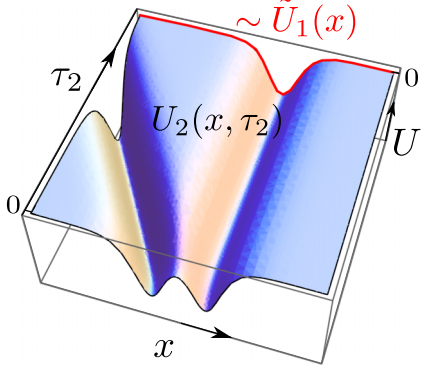

FIG. 3 (color online). For the particular case of $n = 2$, the figure illustrates the transformation of the Schrödinger n-soliton potential U_n into the $(n-1)$-soliton potential $\tilde{U}_{n-1}$ in the limit $\tau_n \to \infty$.

The effect of the limit $\tau_j \to \pm\infty$ results then in the reduction of the reflectionless system $H_n(x;\kappa_1,\tau_1,\ldots,\kappa_n,\tau_n)$ into the reflectionless system $\hat{H}_{n-1}$, where the latter Hamiltonian is given by the set of parameters κ_i and $\hat{\tau}_i$ with $i = 1,\ldots,j-1,j+1,\ldots,n$. It is also easy to check that the application of the limit $\kappa_j \to \kappa_{j+1}$, with j taking one of the values from the set $1,\ldots,n-1$, transforms H_n into H_{n-2}, where the latter reflectionless Hamiltonian is characterized by the parameters κ_i and τ_i with $i = 1,\ldots,j-1,j+1,\ldots,n$. The same effect can be obtained if we apply subsequently two limits, first $\tau_j \to +\infty$ (or $\tau_j \to -\infty$) and then $\hat{\tau}_{j-1} \to -\infty$ (or $\hat{\tau}_{j-1} \to +\infty$), i.e. sent the soliton j and the transformed one with index $j-1$ to infinity in the opposite directions.

Note here that, applying appropriately the described limits with τ_j tending to $+\infty$ or $-\infty$, we can reproduce exactly the phases from (2.2), which correspond to the soliton scattering picture in the n-soliton solution for the KdV equation. Indeed, let us fix index $i = l$, where

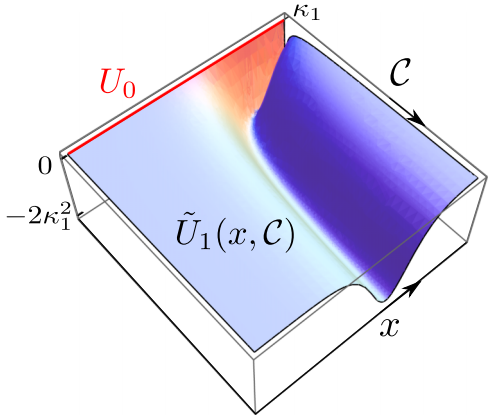

FIG. 4 (color online). As an illustration for the second limit in (3.9), the transformation is shown of the one-soliton potential $\tilde{U}_1(x,\mathcal{C})$ into the zero potential of the free particle case in the limit $\mathcal{C}\to\kappa_1$. Note that in another limit $\mathcal{C}\to\infty$, we have $\tilde{H}_n\to H_n$, but the intertwining operator X_n blows up. Changing X_n for the rescaled operator $\hat{X}_n=\frac{1}{\mathcal{C}}X_n$, we get in the indicated limit the trivial operator $\hat{X}_n\to 1$, as an intertwiner between the two identical copies of the reflectionless Schrödinger Hamiltonian H_n.

$1\le l\le n$. For the sake of generality, assume that $1<l<n$. Now, let us take a limit $\tau_n\to+\infty$. The displaced value of τ_l will be given by the upper sign case of Eq. (3.18) with $i=l$ and $j=n$. Then we send subsequently to $+\infty$ the soliton indexed by $j=n-1$, then $j=n-2$, etc., till $j=l+1$. Repeating the analogous procedure with sending to $-\infty$ first the soliton with $j=1$, then with $j=2$, etc., till $j=l-1$, the resulting changed translation parameter will be given exactly by Eq. (2.2) corresponding to the case $t\to-\infty$ with τ_l^0 changed for our initial value τ_l. The minus sign in the limit $t\to-\infty$ (in comparison with the sign in the limit $\tau_n\to+\infty$) is associated with the minus sign appearing in Eq. (2.1) before the term $4\kappa_k^2 t$.

Considering the pairs of reflectionless Hamiltonians (H_n,H_{n-1}) or $(H_n,\tilde{H}_n)$, the partners of which are related by the first-order Darboux intertwining generators, we shall see below that the limits $\tau_n\to\pm\infty$ induce the transmutation of the type of the supersymmetry, interchanging the cases of the unbroken and broken supersymmetries. On the other hand, the application of the limits $\tau_j\to\pm\infty$ with $j<n$ reduces only the number of the bound states in the partner Hamiltonians but does not change the type of the corresponding supersymmetry of the extended reflectionless system.

The difference of the corresponding supersymmetries in the two cases can be explained by the different nature of the first-order Darboux intertwining generators. In the case of the pairs (H_n,H_{n-1}), the intertwining generators A_n and $A_n^\dagger$ are constructed in terms of the superpotential $\mathcal{W}_n$ [see Eq. (2.10)], for which $\mathcal{W}_n(x)\to\mp\kappa_n$ for $x\to\pm\infty$. This superpotential takes asymptotically the constant values of the opposite signs and is topologically nontrivial. The Witten index for such an extended system takes a nonzero value, and the associated first-order supersymmetry (see the next section) is unbroken [24,25]. The isospectral partners in the pairs $(H_n,\tilde{H}_n)$ are intertwined by the first-order Darboux generators X_n and $X_n^\dagger$, constructed in terms of the superpotential $\Delta_n(\mathcal{C})$; see Eq. (3.6) with $n-1$ changed for n. Since $\lim_{x\to+\infty}\Delta_n=\lim_{x\to-\infty}\Delta_n=\mathcal{C}$ with $\mathcal{C}^2>\kappa_n^2>0$, the superpotential $\Delta_n(\mathcal{C})$ is topologically trivial, and the corresponding first-order supersymmetry will be broken in correspondence with the zero value of the Witten index.

IV. EXOTIC SUPERSYMMETRY OF REFLECTIONLESS SYSTEMS WITH THE FIRST-ORDER SUPERCHARGES

Consider now an extended 2×2 matrix Hamiltonian $\mathcal{H}=\mathrm{diag}(H,H')$ with H and H' to be reflectionless systems and identify $\Gamma=\sigma_3$ as a $\mathbb{Z}_2$-grading operator. As it was shown in [34], in the general case such a system is characterized by exotic nonlinear supersymmetry with two pairs of supercharges, which are the matrix higher-order derivative operators of the antidiagonal form, constructed from the Darboux-Crum intertwiners. The symmetry structure of $\mathcal{H}$ also has to include two higher-order Lax-Novikov integrals of the subsystems H and H'. Within this class of the extended reflectionless systems, there exist two special families, for which a pair of fermionic integrals are the first-order matrix differential operators of the form $S_a=S_a^\dagger=\sigma_a\mathrm{diag}(D,D^\dagger)$, $a=1,2$, which satisfy the relations $[S_a,\mathcal{H}]=0$ and $\{S_a,S_b\}=2\delta_{ab}(\mathcal{H}+\mathrm{const})$. The operators D and $D^\dagger$ in this case not only intertwine the Hamiltonian operators H and H', but also factorize them, $H=D^\dagger D+\mathrm{const}$ and $H'=DD^\dagger+\mathrm{const}$.[4]

Without loss of generality, one can choose $H=H_n$ to be a reflectionless Hamiltonian with an n-soliton potential. Then there are only three possibilities to choose H' such that H and H' can be related by the intertwining operators of the first order. These possibilities are $H'=H_{n-1}$, $H'=H_{n+1}$, or $H'=\tilde{H}_n(\mathcal{C})$. The trivial case of a free particle, H_0, is exceptional: for it there are only two possibilities, $H'=H_1$ and $H'=H_0$, due to the translation invariance of H_0.

Having this picture in mind, we first consider a class of the extended reflectionless $(2n+1)$-parametric systems composed from isospectral Hamiltonians each having n bound states. It is convenient to shift the Hamiltonian operators for an additive constant term and take

$$\breve{\mathcal{H}}_n=\begin{pmatrix}H_n^{\mathcal{C}} & 0\\ 0 & \tilde{H}_n^{\mathcal{C}}\end{pmatrix} \tag{4.1}$$

[4]The supercharges, which are the higher-order derivative operators, factorize certain polynomials of the partner Hamiltonians in correspondence with relations of the form (2.18).

as the extended Hamiltonian, where $H_n^{\mathcal{C}} = H_n + \mathcal{C}^2$, $\tilde{H}_n^{\mathcal{C}} = \tilde{H}_n + \mathcal{C}^2$. A real constant $\mathcal{C}$ is restricted here by the condition $\mathcal{C}^2 > \kappa_n^2$, and $\tilde{H}_n = \tilde{H}_n(\mathcal{C})$ is the reflectionless system isospectral to H_n but with the parameters τ_i in the n-soliton potential changed for the shifted set of translation parameters $\tilde{\tau}_i$ given by Eq. (3.1). The spectra of the isospectral partner Hamiltonians are

$$\sigma(H_n^{\mathcal{C}}) = \sigma(\tilde{H}_n^{\mathcal{C}}) = \mathcal{C}^2 - \kappa_n^2 \cup \cdots \cup \mathcal{C}^2 - \kappa_1^2 \cup [\mathcal{C}^2, \infty).$$

Each discrete energy level $\mathcal{C}^2 - \kappa_i^2$, $i = 1, \ldots, n$, of the extended system (4.1) as well as the energy level $E = \mathcal{C}^2$ at the edge of the continuous part of the spectrum are doubly degenerate. At the same time, each energy level inside the conduction band $(\mathcal{C}^2, \infty)$ of $\breve{\mathcal{H}}_n$ is fourfold degenerate. The set of the nontrivial integrals of motion (in addition to the trivial integral $\Gamma = \sigma_3$) of the supersymmetric system (4.1) consists of the two matrix differential operators of the first order composed from the Darboux intertwining generators of the form (3.6) (with index $n-1$ changed for n):

$$\breve{\mathcal{S}}_{n,1} = \begin{pmatrix} 0 & X_n \\ X_n^\dagger & 0 \end{pmatrix}, \qquad \breve{\mathcal{S}}_{n,2} = i\sigma_3 \breve{\mathcal{S}}_{n,1}. \tag{4.2}$$

We have also two matrix integrals to be differential operators of the order of $2n$ constructed from the intertwines (3.12):

$$\breve{\mathcal{Q}}_{n,1} = \begin{pmatrix} 0 & \mathcal{Y}_n \\ \mathcal{Y}_n^\dagger & 0 \end{pmatrix}, \qquad \breve{\mathcal{Q}}_{n,2} = i\sigma_3 \breve{\mathcal{Q}}_{n,1}. \tag{4.3}$$

In addition, the system is characterized by the two diagonal matrix integrals constructed from the Lax-Novikov integrals (2.19) of the subsystems, which are the differential operators of the order of $2n+1$:

$$\breve{\mathcal{P}}_{n,1} = \begin{pmatrix} \mathcal{L}_n & 0 \\ 0 & \tilde{\mathcal{L}}_n \end{pmatrix}, \qquad \breve{\mathcal{P}}_{n,2} = \sigma_3 \breve{\mathcal{P}}_{n,1}. \tag{4.4}$$

With the chosen $\mathbb{Z}_2$-grading operator $\Gamma = \sigma_3$, operators (4.2) and (4.3) are identified as the fermionic integrals, and (4.4) are identified as the bosonic generators. They, together with the Hamiltonian $\breve{\mathcal{H}}_n$, generate the exotic superalgebra, whose nonzero (anti)commutation relations are given by

$$\{\breve{\mathcal{S}}_a, \breve{\mathcal{S}}_b\} = 2\delta_{ab}\breve{\mathcal{H}}, \qquad \{\breve{\mathcal{Q}}_a, \breve{\mathcal{Q}}_b\} = 2\delta_{ab}\breve{\mathbb{P}}^2,$$
$$\{\breve{\mathcal{S}}_a, \breve{\mathcal{Q}}_b\} = 2\delta_{ab}\mathcal{C}\breve{\mathbb{P}} + 2\epsilon_{ab}\breve{\mathcal{P}}_1, \tag{4.5}$$

$$[\breve{\mathcal{P}}_2, \breve{\mathcal{S}}_a] = 2i(\breve{\mathcal{H}}\breve{\mathcal{Q}}_a - \mathcal{C}\breve{\mathbb{P}}\breve{\mathcal{S}}_a),$$
$$[\breve{\mathcal{P}}_2, \breve{\mathcal{Q}}_a] = 2i\breve{\mathbb{P}}(\mathcal{C}\breve{\mathcal{Q}}_a - \breve{\mathbb{P}}\breve{\mathcal{S}}_a), \tag{4.6}$$

where $\breve{\mathbb{P}}_n = \prod_{j=1}^n (\breve{\mathcal{H}}_n - \mathcal{C}^2 + \kappa_j^2)$ and to simplify the expressions we omitted the index n in (4.5) and (4.6). Though our construction with the two Schrödinger subsystems $H_n^{\mathcal{C}}$ and $\tilde{H}_n^{\mathcal{C}}$ corresponds to the usual $\mathcal{N} = 2$ supersymmetry generated by the two supercharges $\breve{\mathcal{S}}_{n,a}$ to be matrix differential operators of the first order, we have obtained the exotic supersymmetric structure with the two additional supercharges $\breve{\mathcal{Q}}_{n,a}$ to be the higher-order differential operators. In addition, being the extended reflectionless system, it also possesses two bosonic integrals of motion. The peculiarity of the present exotic supersymmetric structure is that the bosonic integral $\breve{\mathcal{P}}_{n,1}$ commutes with all the other integrals of motion and plays a role of the central charge operator of the nonlinear superalgebra.[5] Another bosonic integral $\breve{\mathcal{P}}_{n,2}$ realizes a rotation of the pairs of the supercharges $\breve{\mathcal{S}}_{n,a}$ and $\breve{\mathcal{Q}}_{n,a}$ by means of the commutation relations (4.6) with the Hamiltonian-dependent structure coefficients.

Since the doublet of the ground states of $\breve{\mathcal{H}}_n$ has positive energy $\mathcal{C}^2 - \kappa_n^2 > 0$, the first-order supercharges $\breve{\mathcal{S}}_{n,a}$ do not annihilate them either, and the $\mathcal{N} = 2$ Lie subsuperalgebra generated by $\breve{\mathcal{S}}_{n,a}$ and $\breve{\mathcal{H}}_n$ corresponds to the phase of the *broken* supersymmetry. At the same time, according to Eq. (2.20), the doublet of the ground states is annihilated by the bosonic integrals $\breve{\mathcal{P}}_{n,a}$. Because of the second relation from (4.5), they are also annihilated by the higher-order supercharges $\breve{\mathcal{Q}}_{n,a}$. One can conclude therefore that the obtained exotic nonlinear $\mathcal{N} = 4$ supersymmetry of the extended reflectionless system $\breve{\mathcal{H}}_n$ is *partially broken*.

Let us apply now the limit $\mathcal{C}^2 \to \kappa_n^2$, associated with the soliton scattering, to the system $\breve{\mathcal{H}}_n$. For the sake of definiteness, let us assume that $\mathcal{C}$ is positive and consider the limit $\mathcal{C} \to \kappa_n$, which corresponds to the limit $\tilde{\tau}_n \to -\infty$ for the subsystem $\tilde{H}_n^{\mathcal{C}}$. In this limit, the Hamiltonian (4.1) and integrals of motion are transformed into

$$\mathcal{H}_n = \begin{pmatrix} H_n^{\kappa_n} & 0 \\ 0 & H_{n-1}^{\kappa_n} \end{pmatrix}, \qquad \mathcal{S}_{n,1} = \begin{pmatrix} 0 & A_n \\ A_n^\dagger & 0 \end{pmatrix}, \tag{4.7}$$

$$\mathcal{Q}_{n,1} = \begin{pmatrix} 0 & \mathcal{B}_n \\ \mathcal{B}_n^\dagger & 0 \end{pmatrix}, \qquad \mathcal{P}_{n,1} = \begin{pmatrix} \mathcal{L}_{2n+1} & 0 \\ 0 & H_{n-1}^{\kappa_n}\mathcal{L}_{2n-1} \end{pmatrix}, \tag{4.8}$$

and the integrals with index $a = 2$ are obtained by the same rule as in (4.2)–(4.4), where the notations $H_n^{\kappa_n} = H_n + \kappa_n^2$ and $H_{n-1}^{\kappa_n} = H_{n-1} + \kappa_n^2$ are used. To obtain the limit we have taken into account the relations (3.7), (3.12), and (2.16). The Hamiltonian $\mathcal{H}_n$ and its integrals of motion generate the nonlinear superalgebra of the form similar to (4.5) and (4.6), but with corresponding changes of the operators on the right-hand sides, and with the $\mathcal{C}$ changed for κ_n.

[5] This is not so in a general case of the extended system composed from the two n-soliton Schrödinger subsystems; see Ref. [34].

Note that the lower matrix element in the integral $\mathcal{P}_{n,1}$ (and, similarly, in $\mathcal{P}_{n,2}$) is factorized into the subsystem's Hamiltonian $H_{n-1}^{\kappa_n}$ and the corresponding Lax-Novikov integral. The multiplicative factor $H_{n-1}^{\kappa_n}$ could be omitted there without changing the property of commutativity of the diagonal matrix operators with the Hamiltonian $\mathcal{H}_n$. However, this would change the property that the upper and lower matrix elements in the integrals $\mathcal{P}_{n,a}$ are the differential operators of the same order of $2n+1$ and, as a consequence, would complicate the form of the superalgebra.

In spite of a similar form of the superalgebra (with $\mathcal{C}$ changed for κ_n), the superextended system we have here is essentially different from the previous one. Indeed, the system $\mathcal{H}_n$, unlike the $\breve{\mathcal{H}}_n$, possesses now the nondegenerate ground state of zero energy, which corresponds to the lowest bound state of the upper subsystem $H_n^{\kappa_n}$. This state is annihilated by all four supercharges and the two bosonic integrals, and the exotic nonlinear supersymmetry we have here corresponds to the *unbroken* phase. Therefore, the limit we considered provokes the transmutation of the partially broken exotic supersymmetry into the unbroken one.

Also, there exists a limit, associated with the soliton scattering, which transmutes the exotic nonlinear supersymmetry from the unbroken phase into the partially broken exotic supersymmetry. To see this, we apply to the system (4.2) and (4.8) the limit $\tau_n \to \infty$, which corresponds to sending the soliton with index n in the subsystem $H_n^{\kappa_n}$ to infinity. We find then with the help of (3.15) and (3.16) that

$$\mathcal{H}_n \underset{\tau_n\to\infty}{\to} \breve{\mathcal{H}}_{n-1}^{\diamond}, \qquad \mathcal{S}_{n,a} \underset{\tau_n\to\infty}{\to} \breve{\mathcal{S}}_{n-1,a}^{\diamond}, \tag{4.9}$$

$$\mathcal{P}_{n,a} \underset{\tau_n\to\infty}{\to} \breve{\mathcal{H}}_{n-1}^{\diamond}\breve{\mathcal{P}}_{n-1,a}^{\diamond},$$

$$\mathcal{Q}_{n,a} \underset{\tau_n\to\infty}{\to} -\breve{\mathcal{H}}_{n-1}^{\diamond}\breve{\mathcal{Q}}_{n-1,a}^{\diamond} + 2\kappa_n \breve{\mathbb{P}}_{n-1}^{\diamond}\breve{\mathcal{S}}_{n-1,a}^{\diamond}. \tag{4.10}$$

Here we have used the notation $F^{\diamond} = \sigma_2 F \sigma_2$, which corresponds to a unitary transformation between the matrix operators

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad F^{\diamond} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix},$$

and imply that the operators indexed by $n-1$ are given by the same expressions as the operators associated with $\breve{\mathcal{H}}_n$, but with the parameter $\mathcal{C}$ changed in the structure of the latter operators for $\mathcal{C} = \kappa_n$. As a consequence, we also obtain a four-term chain of the limits

$$\mathcal{H}_n \underset{\tau_n\to\infty}{\to} \breve{\mathcal{H}}_{n-1}^{\diamond} \underset{\kappa_n\to\kappa_{n-1}}{\to} \mathcal{H}_{n-1}^{\diamond} \underset{\tau_{n-1}\to\infty}{\to} \breve{\mathcal{H}}_{n-2}. \tag{4.11}$$

Note that the multiplicative factor $\breve{\mathcal{H}}_{n-1}^{\diamond}$ in the limit of the operators $\mathcal{P}_{n,a}$ and $\mathcal{Q}_{n,a}$ in (4.10) corresponds to a reduction of the order of the integrals that is related with the loss of the one eigenvalue of zero energy in comparison with the spectrum of the system $\mathcal{H}_n$.

V. TRANSPARENT DIRAC SYSTEMS

We have discussed the Darboux-Crum transformations, the exotic supersymmetric structure based on them, and transmutations of supersymmetry in the reflectionless systems described by the 2×2 matrix second-order Schrödinger Hamiltonian operators. One can take one of the two first-order Hermitian supercharges appearing in these second-order systems and consider it as a first-order matrix Hamiltonian for the $(1+1)$-dimensional Dirac system. We can identify then the Darboux-Crum generators, which intertwine such reflectionless first-order matrix Hamiltonians. This opens a possibility to investigate exotic supersymmetry and its transmutations in the transparent Dirac systems.

Let us take the first-order supercharge $\breve{\mathcal{S}}_{n,1}$ from (4.2) and identify it as the Dirac Hamiltonian: $\breve{H}_n^D \equiv \breve{\mathcal{S}}_{n,1}$. This system corresponds to the $(1+1)$-dimensional Dirac particle in a scalar potential $\Delta_n(x) = \Omega_n - \tilde{\Omega}_n + \mathcal{C}$ with asymptotic behavior $\Delta_n(x) \to \mathcal{C}$ for $x \to \pm\infty$. Because of the relation of commutativity $[\breve{\mathcal{S}}_{n,1}, \breve{\mathcal{P}}_{n,1}] = 0$, the potentials of this form correspond to the solutions of the multikink-antikink type for the stationary mKdV hierarchy [35].

The Dirac Hamiltonian $\breve{H}_n^D$ has $2n$ bound states, and its spectrum is symmetric:

$$\sigma(\breve{H}_n^D) = (-\infty,-\mathcal{C}] \cup \breve{\mathcal{E}}_1^- \cup \cdots \cup \breve{\mathcal{E}}_n^- \cup \breve{\mathcal{E}}_n^+ \cup \cdots \cup \breve{\mathcal{E}}_1^+ \cup [\mathcal{C},\infty), \tag{5.1}$$

where $\breve{\mathcal{E}}_i^{\pm} = \pm\sqrt{\mathcal{C}^2 - \kappa_i^2}$, $i = 1,\ldots,n$, and semi-infinite intervals $[\mathcal{C},\infty)$ and $(-\infty,-\mathcal{C}]$ correspond to the doubly degenerate continuous parts of the spectrum. In the limit $\mathcal{C} \to \kappa_n$, we have $\breve{H}_n^D \to H_n^D = \mathcal{S}_{n,1}$, where $\mathcal{S}_{n,1}$ is defined in (4.7). A scalar potential takes here the form $\mathcal{W}_n(x) = \Omega_n - \Omega_{n-1}$, with $\mathcal{W}_n(x) \to \mp\kappa_n$ for $x \to \pm\infty$. The potentials of this form are, again, the solutions of the kink (or antikink) type for the stationary mKdV hierarchy due to the relation $[\mathcal{S}_{n,1}, \mathcal{P}_{n,1}] = 0$. The spectrum of the Dirac Hamiltonian H_n^D has $2n-1$ bound states, including one bound state of zero energy:

$$\sigma(H_n^D) = (-\infty,-\kappa_n] \cup \mathcal{E}_1^- \cup \cdots \cup \mathcal{E}_{n-1}^- \cup 0 \cup \mathcal{E}_{n-1}^+ \cup \cdots \cup \mathcal{E}_1^+ \cup [\kappa_n,\infty), \tag{5.2}$$

where $\mathcal{E}_i^{\pm}=\pm\sqrt{\kappa_n^2-\kappa_i^2}$, $i=1,\ldots,n-1$. The two discrete energy levels $\breve{\mathcal{E}}_n^-$ and $\breve{\mathcal{E}}_n^+$ of the system $\breve{H}_n^D$ merge in the limit $\mathcal{C}\to\kappa_n$ and transform into a nondegenerate zero energy level of the bound state for the system H_n^D.

A. First-order matrix Darboux intertwiners for Dirac systems

Let us return to the identity (3.13),

$$A_n(x,\tau_i)X_{n-1}(x,\tau_i,\mathcal{C})=X_n(x,\tau_i,\mathcal{C})A_n(x,\tilde{\tau}_i),\tag{5.3}$$

where $X_n(x,\tau_i,\mathcal{C})=\frac{d}{dx}+\Delta_n(x,\tau_i,\mathcal{C})$ and

$$\Delta_n(x,\tau_i,\mathcal{C})=\Omega_n(x,\tau_i)-\Omega_n(x,\tilde{\tau}_i)+\mathcal{C},$$
$$\tilde{\tau}_i=\tau_i-\varphi_i(\mathcal{C}),\qquad \varphi_i(\mathcal{C})=\frac{1}{2\kappa_i}\ln\frac{\mathcal{C}+\kappa_i}{\mathcal{C}-\kappa_i}.\tag{5.4}$$

If in (5.3) we change $\tau_i\to\tau_i+\varphi_i(\mathcal{C})$, then make a replacement $\mathcal{C}\to-\mathcal{C}$, and take into account that $\varphi_i(-\mathcal{C})=-\varphi_i(\mathcal{C})$ and that X_n satisfies the relation $X_n(x,\tau_i-\varphi_i(\mathcal{C}),-\mathcal{C})=-X_n^{\dagger}(x,\tau_i,\mathcal{C})$, we obtain the identity

$$A_n(x,\tilde{\tau}_i)X_{n-1}^{\dagger}(x,\tau_i,\mathcal{C})=X_n^{\dagger}(x,\tau_i,\mathcal{C})A_n(x,\tau_i).\tag{5.5}$$

By using the notations $A_n\equiv A_n(x,\tau_i)$, $\tilde{A}_n(\mathcal{C})\equiv A_n(\tilde{\tau}_i)$, and $X_n(\mathcal{C})\equiv X_n(x,\tau_i,\mathcal{C})$, Eqs. (5.3)–(5.5) and their Hermitian conjugate give us the relations

$$A_nX_{n-1}=X_n\tilde{A}_n,\qquad A_n^{\dagger}X_n=X_{n-1}\tilde{A}_n^{\dagger},\tag{5.6}$$

$$\tilde{A}_nX_{n-1}^{\dagger}=X_n^{\dagger}A_n,\qquad \tilde{A}_n^{\dagger}X_n^{\dagger}=X_{n-1}^{\dagger}A_n^{\dagger}.\tag{5.7}$$

Using these relations, we can define the intertwining operator between the Dirac Hamiltonians $\breve{H}_n^D$ and $\breve{H}_{n-1}^D$, which also is the intertwining operator between the extended (supersymmetric) Schrödinger Hamiltonians $\breve{\mathcal{H}}_n$ and $\breve{\mathcal{H}}_{n-1}$:

$$\breve{\mathcal{A}}_n=\begin{pmatrix}A_n & 0\\ 0 & \tilde{A}_n\end{pmatrix},\qquad \breve{\mathcal{A}}_n\breve{H}_{n-1}^D=\breve{H}_n^D\breve{\mathcal{A}}_n,$$
$$\breve{\mathcal{A}}_n\breve{\mathcal{H}}_{n-1}=\breve{\mathcal{H}}_n\breve{\mathcal{A}}_n.\tag{5.8}$$

In the limit $\mathcal{C}\to\kappa_n$, the relations in (5.6) are transformed into the trivial identity $A_nX_{n-1}(\kappa_n)=A_nX_{n-1}(\kappa_n)$, and the relation

$$A_n^{\dagger}A_n=X_{n-1}(\kappa_n)X_{n-1}^{\dagger}(\kappa_n)=H_{n-1}+\kappa_n^2,\tag{5.9}$$

where we have used the limits (3.7). These identities allow us to construct a new operator of intertwining between the Dirac systems H_n^D and $\breve{H}_{n-1}^D$ and between the superextended Schrödinger Hamiltonians $\mathcal{H}_n$ and $\breve{\mathcal{H}}_{n-1}$:

$$\mathcal{A}_n=\begin{pmatrix}A_n & 0\\ 0 & X_{n-1}(\kappa_n)\end{pmatrix},\qquad \mathcal{A}_n\breve{H}_{n-1}^D(\kappa_n)=H_n^D\mathcal{A}_n,$$
$$\mathcal{A}_n\breve{\mathcal{H}}_{n-1}(\kappa_n)=\mathcal{H}_n\mathcal{A}_n,\tag{5.10}$$

where we indicated a dependence of the corresponding operators on $\kappa_n=\mathcal{C}$.

This construction corresponds here to the Darboux transformations for reflectionless Dirac systems and, particularly, gives us a possibility to construct analytically the states of H_n^D and $\breve{H}_n^D$ in terms of the eigenstates $\breve{\Phi}_0$ of the matrix operator $\breve{H}_0^D=-\sigma_2p+\sigma_1\mathcal{C}$:

$$\breve{H}_0^D(\mathcal{C})=\begin{pmatrix}0 & \frac{d}{dx}+\mathcal{C}\\ -\frac{d}{dx}+\mathcal{C} & 0\end{pmatrix},\tag{5.11}$$

which corresponds to the Hamiltonian of the free massive Dirac particle. The eigenstates $\breve{\Phi}_n$ of $\breve{H}_n^D$ can be presented in the form $\breve{\Phi}_n=\breve{\mathcal{A}}_n\breve{\mathcal{A}}_{n-1}\ldots\breve{\mathcal{A}}_1\breve{\Phi}_0$, while the eigenstates of H_n^D are constructed in the form $\Phi_n=\mathcal{A}_n\breve{\mathcal{A}}_{n-1}\breve{\mathcal{A}}_{n-2}\ldots\breve{\mathcal{A}}_1\breve{\Phi}_0$ in terms of the eigenstates $\breve{\Phi}_0$ of the Dirac Hamiltonian $\breve{H}_0^D(\kappa_n)=-\sigma_2p+\sigma_1\kappa_n$. The explicit forms of the scattering states and $2n$ bound states of the $\breve{H}_n^D$ are given by

$$\breve{\Phi}_n^{\epsilon}(\breve{\mathcal{E}}^{\pm}(k^2))=\begin{pmatrix}\Psi_n^{\epsilon}(k^2)\\ \pm\sqrt{\frac{\mathcal{C}-i\epsilon k}{\mathcal{C}+i\epsilon k}}\tilde{\Psi}_n^{\epsilon}(k^2)\end{pmatrix},$$
$$\breve{\Phi}_n(\breve{\mathcal{E}}_i^{\pm})=\begin{pmatrix}\Psi_n(-\kappa_i^2)\\ \pm\tilde{\Psi}_n(-\kappa_i^2)\end{pmatrix},\tag{5.12}$$

where $\breve{H}_n^D\breve{\Phi}_n(\breve{\mathcal{E}})=\breve{\mathcal{E}}\breve{\Phi}_n(\breve{\mathcal{E}})$, $\breve{\mathcal{E}}^{\pm}(k^2)=\pm\sqrt{\mathcal{C}^2+k^2}$, $\breve{\mathcal{E}}_i^{\pm}=\pm\sqrt{\mathcal{C}^2-\kappa_i^2}$, $i=1,\ldots,n$, Ψ_n are Schrödinger eigenstates defined in (2.8), and the parameter $\epsilon=\pm1$ corresponds to the two possible directions in which the waves can propagate. The two discrete energy levels $\breve{\mathcal{E}}_n^{\pm}=\pm\sqrt{\mathcal{C}^2-\kappa_n^2}$ merge in the limit $\mathcal{C}\to\kappa_n$, and two corresponding eigenstates of $\breve{H}_n^D$ reduce to the unique state of zero energy of the Dirac Hamiltonian H_n^D:

$$\breve{\Phi}_n(\breve{\mathcal{E}}_n^{\pm})=\begin{pmatrix}\Psi_n(-\kappa_n^2)\\ \pm\tilde{\Psi}_n(-\kappa_n^2)\end{pmatrix}\to\Phi_n(0)=\begin{pmatrix}\Psi_n(-\kappa_n^2)\\ 0\end{pmatrix}.\tag{5.13}$$

B. Exotic supersymmetry of reflectionless Dirac systems

The matrix operator $\breve{\mathcal{P}}_{n,1}$ and the Dirac Hamiltonian $\breve{\mathcal{H}}_n^D$ correspond to the Lax pair for the nth member of the stationary mKdV hierarchy, and the scalar Dirac potential $\Delta_n(x)$ is identified as the corresponding soliton (multikink-antikink type) solution. Since $[\breve{\mathcal{P}}_{n,1},\breve{\mathcal{H}}_n^D]=0$, the $\breve{\mathcal{P}}_{n,1}$ is a nontrivial integral for the Dirac system $\breve{H}_n^D$. It is the Darboux-dressed momentum operator of the free Dirac massive particle (5.11). The interesting point is that for the

reflectionless Dirac system $\breve{H}_n^D$ one can identify an additional integral of motion $\breve{\Gamma}$, which satisfies the identity $\breve{\Gamma}^2 = 1$ and anticommutes with $\breve{\mathcal{P}}_{n,1}$. As a consequence, the reflectionless Dirac system $\breve{H}_n^D$ can be characterized by the proper exotic nonlinear supersymmetry. Indeed, consider the operator $\breve{\Gamma} = \mathcal{R}\sigma_3$, where $\mathcal{R}$ is the operator of reflection in x, τ_i, and $\mathcal{C}$, which satisfies the relations $\mathcal{R}z = -z\mathcal{R}$, $\mathcal{R}^2 = 1$, where $z = x, \tau_i$, or $\mathcal{C}$. Because of the relations $[\breve{\Gamma}, \breve{\mathcal{H}}_n^D] = 0$ and $\{\breve{\Gamma}, \breve{\mathcal{P}}_{n,1}\} = 0$, the $\breve{\mathcal{H}}_n^D$ and $\breve{\mathcal{P}}_{n,1}$ are identified as bosonic and fermionic operators, respectively. They generate a nonlinear $\mathcal{N} = 1$ superalgebra

$$[\breve{\mathcal{P}}_{n,1}, \breve{\mathcal{H}}_n^D] = 0, \quad \{\breve{\mathcal{P}}_{n,1}, \breve{\mathcal{P}}_{n,1}\} = 2\mathbb{P}_{2(2n+1)}(\breve{\mathcal{H}}_n^D), \quad (5.14)$$

where

$$\mathbb{P}_{2(2n+1)}(\breve{\mathcal{H}}_n^D) \equiv ((\breve{\mathcal{H}}_n^D)^2 - \mathcal{C}^2)\prod_{j=1}^{n}((\breve{\mathcal{H}}_n^D)^2 - (\mathcal{C}^2 - \kappa_j^2))^2. \quad (5.15)$$

The $2(n+1)$ zeros of the polynomial in $\mathcal{H}_n^D$ operator (5.15) correspond to the energies of the singlet states of the reflectionless Dirac system, where $\breve{\mathcal{E}}_i^{\pm} = \pm\sqrt{\mathcal{C}^2 - \kappa_i}$, $i = 1, \ldots, n$, are the energies of the bound states, while $\pm\mathcal{C}$ correspond to the two singlet states at the edges of the continuous parts of the spectrum.[6] In accordance with (2.8), the left- and right-moving waves in (5.12) of the scattering sector, which correspond to doubly degenerate energy levels $\breve{\mathcal{E}}^{\pm}(k^2)$ of $\breve{H}_n^D$, are distinguished by the supercharge $\breve{\mathcal{P}}_{n,1}$: they are its eigenstates of the opposite sign eigenvalues. By supplementing the integral $\breve{\mathcal{P}}_{n,1}$ with a (nonlocal) integral $\breve{\mathcal{P}}_{n,2} = i\breve{\Gamma}\breve{\mathcal{P}}_{n,1}$, the $\mathcal{N} = 1$ exotic nonlinear supersymmetry of the reflectionless Dirac system $\breve{H}_n^D$ can be extended to $\mathcal{N} = 2$: $\{\breve{\mathcal{P}}_{n,a}, \breve{\mathcal{P}}_{n,b}\} = 2\delta_{ab}\mathbb{P}_{2(2n+1)}(\breve{\mathcal{H}}_n^D)$.

Applying the limit $\mathcal{C} \to \kappa_n$, we identify the proper exotic supersymmetric structure of H_n^D. In this case, the zero energy eigenstate (5.13) of H_n^D is also the zero mode of the supercharge $\mathcal{P}_{n,1}$. In both Dirac reflectionless systems $\breve{H}_n^D$ and H_n^D, the supercharges detect all the nondegenerate eigenvalues of the Hamiltonians by annihilating the corresponding eigenstates, which are the bound states and the states at the edges of the continuous parts of the spectra. Since the zero energy eigenvalue belongs to the spectrum of H_n^D but is not present in the spectrum of $\breve{H}_n^D$, the proper exotic supersymmetry of the Dirac system $\breve{H}_n^D$ is of the broken nature, while that of H_n^D corresponds to the unbroken phase. In correspondence with the second relation from (4.9), the limit $\tau_n \to \infty$ applied to the Dirac system H_n^D with the unbroken proper exotic supersymmetry will produce the system $\breve{H}_{n-1}^{\diamond D} = \breve{\mathcal{S}}_{n-1,1}^{\diamond}$ [see Eq. (4.9)], characterized by the broken exotic supersymmetry.

[6]Besides a bound state, each double root $\breve{\mathcal{E}}_i^{\pm}$, $i = 1, \ldots, n$, of the polynomial on the left-hand side of (5.15) corresponds also to a nonphysical eigenstate of $\breve{\mathcal{H}}_n^D$.

VI. DISCUSSION AND OUTLOOK

We have considered the two related families of the $(1+1)$D Dirac reflectionless systems. Each such system corresponds to a fermion in a background of a multisoliton solution (of the kink or kink-antikink type) of the mKdV equation. In one of these two families, the n-soliton potential $V^D(x) = \Delta_n(x)$ or $-\Delta_n(x)$, where $\Delta_n(x) = \Delta_n(x; \kappa_1, \tau_1, \ldots, \kappa_n, \tau_n, \mathcal{C})$, $\mathcal{C}^2 > \kappa_n^2$, is $(2n+1)$ parametric, while in the second family the potential $V^D(x)$ is $2n$ parametric and corresponds to the function $\mathcal{W}_n(x)$ or $-\mathcal{W}_n(x)$, where $\mathcal{W}_n(x) = \mathcal{W}_n(x; \kappa_1, \tau_1, \ldots, \kappa_n, \tau_n)$. From the viewpoint of the associated extended Schrödinger systems, whose matrix 2×2 Hamiltonians are given by a square of the corresponding Dirac Hamiltonian $H^D = i\sigma_2\frac{d}{dx} + \sigma_1 V^D(x)$, the Dirac potential $V^D(x)$ is a superpotential. The peculiarity of the considered reflectionless families is that in the case of the supersymmetric Schrödinger systems, in addition to the two first-order supercharges H^D and $i\sigma_3 H^D$, they are characterized by the two supercharges to be matrix differentials operators of the order of $2n$. Furthermore, they possess two nontrivial bosonic integrals to be differential operators of the order of $2n+1$, which are constructed from the Lax-Novikov integrals of the Schrödinger subsystems. One of these two bosonic integrals is a central charge of the exotic nonlinear superalgebra. The same higher-order central charge can be identified as the supercharge (a fermionic generator) of the proper exotic nonlinear supersymmetry of the reflectionless Dirac system. In the case of $V^D(x) = \pm\Delta_n(x)$, the exotic nonlinear supersymmetries of the Schrödinger and Dirac systems are spontaneously broken, and the quantity $(\mathcal{C}^2 - \kappa_n^2) > 0$ measures the scale of the breaking. The choices $V^D(x) = \pm\mathcal{W}_n(x)$ correspond, on the other hand, to the unbroken exotic supersymmetries. The interesting point is that there exists a limit procedure, admitting the interpretation in the context of a soliton scattering, which relates the two indicated families of the exotic supersymmetric reflectionless systems. One can define a kind of a topological charge by a relation

$$q = \frac{1}{2|V_0^D|}\int_{-\infty}^{\infty} dx \frac{dV^D(x)}{dx},$$

where $V_0^D = \lim_{x\to+\infty} V^D(x)$. The case of the broken supersymmetry with the kink-antikink type potential $V^D(x) = \pm\Delta_n(x)$ is characterized then by $q = 0$, while the cases of the kink, $V^D(x) = -\mathcal{W}_n(x)$, and antikink, $V^D(x) = \mathcal{W}_n(x)$, type potentials of the unbroken exotic supersymmetries correspond to $q = +1$ and $q = -1$, respectively. The quantity $2|V_0^D|$ gives the gap that separates the upper and lower continuous bands in the spectrum of the Dirac systems and can be treated as a doubled mass parameter of a fermion in an external scalar potential. The mentioned supercharge of the Dirac system annihilates all its nondegenerate energy states and, being the

Darboux-dressed momentum operator of the free Dirac particle (zero-soliton case), distinguishes the left- and right-moving eigenstates corresponding to the doubly degenerate energy values in the continuum bands of the spectrum.

The described transparent potentials $V^D(x)$ appear in many physical applications in the form of stationary solutions for inhomogeneous fermion condensates. Such self-consistent condensates are described by the equations

$$(i\partial - V^D)\psi_\alpha = 0, \qquad V^D = -g^2 \sum_{\alpha=1}^{N} \sum_{\text{occ}} \bar{\psi}_\alpha \psi_\alpha. \quad (6.1)$$

Here the first equation with a generalized flavor index $\alpha = 1, \ldots, N$ represents a system of $(1+1)$D Dirac equations, the $\sum_{\alpha=1}^{N}$ corresponds to summation in degenerate states, and $\sum_{\text{occ}}$ corresponds to a sum over the completely filled Dirac sea plus a sum over bound states, which usually are partially occupied. Equations (6.1) appear particularly in superconductivity, in the Gross-Neveu model, and in the physics of conducting polymers. A famous method of solution of (6.1) was realized by Dashen, Hasslacher, and Neveu in [3], where this system of equations was rewritten in terms of the scattering data for Schrödinger potentials $U_\pm = (V^D)^2 \pm \frac{d}{dx} V^D - (V_0^D)^2$, and as a result it was shown that the reflection coefficient for both potentials $U_\pm$ has to be equal to zero. For some applications of this result, see [55–59]. Using the ideas of supersymmetry, this picture is equivalent to the search of the first-order operators D and $D^\dagger$, which intertwine and factorize corresponding Schrödinger reflectionless Hamiltonians, $H_+ = DD^\dagger - E_0$ y $H_- = D^\dagger D - E_0$. As we have shown, there are only two situations where such a factorization is possible.

(i) When H_+ and H_- are completely isospectral, the V^D corresponds to the Dirac potentials characterized by the topological charge $q = 0$, which are given by inhomogeneous condensates $\pm\Delta_n$ with asymptotic behavior $\Delta_n \to \mathcal{C}$ for $x \to \pm\infty$.

(ii) In the other possible case, the spectra of H_+ and H_- are different in one bound state only, and the inhomogeneous condensate takes here the form $V^D = -\mathcal{W}_n$ or $V^D = \mathcal{W}_n$, where $\mathcal{W}_n \to \mp\kappa_n$ for $x \to \pm\infty$, and the topological charge q takes the values $+1$ or -1.

On the other hand, the occupation fraction for each nondegenerate state defines the energy of the bound states. Using the method of resolvent, Feinberg showed in [4] that for all static condensates the following equality is valid:

$$\nu_i = \frac{2}{\pi} \cot^{-1}\left(\frac{\kappa_i}{\sqrt{(V_0^D)^2 - \kappa_i^2}}\right), \qquad i = 1, \ldots n, \quad (6.2)$$

where ν_i can take the values $\nu_i = 0, \frac{1}{N}, \ldots \frac{N-1}{N}, 1$. This result was reproduced in [58] for complex kinks in the context of the Bogoliubov–de Gennes and chiral Gross-Neveu systems.

The case $N = 1$, $\nu = 0, 1$ corresponds here to the superconductivity. With these restrictions, the topologically trivial homogeneous condensate is possible, $V^D = \pm\Delta_0 = \pm V_0^D$, $\nu_1 = 1$ (free Dirac massive particle), as well as the topologically nontrivial inhomogeneous condensate, $V^D = \pm\mathcal{W}_1$, $\nu_1 = 0$, $\kappa_1 = V_0^D$.

The case $N = 2$, $\nu = 0, 1/2, 1$, corresponds to polymer conductors in the context of the Takayama–Lin-Liu–Maki model [8]; in addition to $V^D = \pm\Delta_0, \pm\mathcal{W}_1$, also the case $\nu_1 = 1/2$, $V^D = \pm\Delta_1$, $\kappa_1 = \frac{1}{\sqrt{2}}|V_0^D|$ is possible. This last solution is known as a polaron. The other topological solution, which is kink + polaron (or antikink + polaron), corresponds to $V^D = \mp\mathcal{W}_2$ ($\kappa_1 = \frac{1}{\sqrt{2}}|V_0^D|$ and $\kappa_2 = |V_0^D|$).

In the 't Hooft limit $N \to \infty$, the κ_i can take any value in the interval $0 \le \kappa_i \le |V_0^D|$, that makes it possible to have any stationary soliton solution. So, we see that for the Gross-Neveu model, the Darboux transformations provide a general method to generate real inhomogeneous condensates for (6.1).

Equations (5.8) and (5.10) allow us to obtain a supersymmetric system described by the extended first-order matrix Hamiltonian composed from the two Dirac Hamiltonians. In such a way, we can get two different families of the extended systems. The first one realizes the unbroken exotic supersymmetry and is given by the Hamiltonian of the form

$$\mathcal{H}^D = \begin{pmatrix} H_n^D & \mathbf{0} \\ \mathbf{0} & \breve{H}_{n-1}^D \end{pmatrix}. \quad (6.3)$$

The matrix integrals for (6.3) given by the first-order differential operators are

$$\mathcal{S}_1^D = \begin{pmatrix} \mathbf{0} & \mathcal{A}_n \\ \mathcal{A}_n^\dagger & \mathbf{0} \end{pmatrix}, \qquad \mathcal{S}_2^D = i\Sigma_3 \mathcal{S}_1^D, \quad (6.4)$$

where Σ_3 is a 4×4 diagonal matrix of the form $\Sigma_3 = \text{diag}(1_2, -1_2)$ with 1_2 the unit 2×2 matrix. Another family is given by the Hamiltonian of the form

$$\breve{\mathcal{H}}^D = \begin{pmatrix} \breve{H}_n^D & \\ \mathbf{0} & \breve{H}_{n-1}^D \end{pmatrix}, \quad (6.5)$$

and its analogous integrals are

$$\breve{\mathcal{S}}_1^D = \begin{pmatrix} \mathbf{0} & \breve{\mathcal{A}}_n \\ \breve{\mathcal{A}}_n^\dagger & \mathbf{0} \end{pmatrix}, \qquad \breve{\mathcal{S}}_2^D = i\Sigma_3 \breve{\mathcal{S}}_1^D. \quad (6.6)$$

The grading operator $\Gamma = \Sigma_3$ identifies the extended Dirac Hamiltonians to be bosonic generators, while (6.4) and (6.6) are identified as the fermionic generators. Then we find that the indicated operators satisfy the nonlinear supersymmetry relations to be of the order of 2 in the

corresponding Hamiltonians: $\{\mathcal{S}^D_a, \mathcal{S}^D_b\} = 2\delta_{ab}(\mathcal{H}^D)^2$, and $\{\check{\mathcal{S}}^D_a, \check{\mathcal{S}}^D_b\} = 2\delta_{ab}((\check{\mathcal{H}}^D)^2 - \mathcal{C}^2 + \kappa_n^2)$. Besides, in each of the two cases, there exist bosonic integrals to be the matrix differential operators of the order of $2n+1$ and fermionic integrals of the order of $2n$.

Also, it is possible to construct supersymmetric Dirac type systems with nonlinear superalgebraic relations of the form $\{S, S\} = 2f((\mathcal{H}^D)^2)$, where f is a polynomial, by taking in extended Hamiltonian $\mathcal{H}^D$ a pair of reflectionless Dirac Hamiltonians with distinct scattering data. The picture has to be similar to that obtained in Ref. [34] for the reflectionless Schrödinger systems.

We are going to present the detailed investigation of such supersymmetric pictures with extended Dirac Hamiltonians elsewhere.

Note also that the last relations in (5.8) and (5.10) can be used to construct further supersymmetric extensions of the reflectionless Schrödinger systems, in particular, given by 4×4 matrix Hamiltonians.

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Capítulo 5

Jerarqía de ecuaciones de Korteweg-de Vries modificadas, transformación de Miura-Darboux y condensados estacionarios auto-consistentes para el modelo de Gross-Neveu

Las soluciones de la Jerarquía de mKdV han sido conectadas con modelos de interacción no lineal entre fermiones. El metodo supersimetrico nos a permitido construir familias infinitas de Hamiltonianos de Schrödinger extendidos con una integral central, cuando una de las integrales fermiónicas de dichas supersimetría es de orden uno, es posible asociar la integral fermiónica y el elemento central del superálgebra con la formulación de par de Lax de la jerarquía de ecuaciones inhomogénea y estacionarias de Korteweg-de Vries modificadas.

Permitiendonos transportar el cuadro exactamente soluble desde los operadores de Schrödinger a operadores de Dirac en 1+1D o de Bogoliubov-de Gennes con una integral de Lax-Novikov, siendo posible entender estos operadores como raices cuadradas del Hamiltoniano de Schrödinger extendido. Acontinuación se muestra como construir las soluciones en forma de condensados escalares estacionarios más generales para modelo de Gross-Neveu.

La jerarquía inhomogénea de ecuaciones KdV modificadas (mKdV)

$$\hat{f}_{n+1} = \frac{dv}{ds} \tag{5.1}$$

es definida como sigue

$$\hat{f}_l = -\frac{i}{2}\hat{f}'_{l-1} + v\,\hat{g}_l, \tag{5.2}$$

$$\hat{g}_l = i\int \left(v\hat{f}_{l-1} - v\,\hat{f}_{l-1}\right) dx + c_l \tag{5.3}$$

$$\hat{f}_{-1} = 0, \hat{f}_0 = v(x), \quad \hat{g}_0 = c_0 = 1, \quad c_{2j+1} = 0 \tag{5.4}$$

los primeros casos de la jerarqía de ecuaciones en el caso estacionario $\hat{f}_{n+1}(x) = 0$ son

$\hat{f}_{-1}(x) = 0,$

$\hat{f}_0(x) = v(x) = 0,$

$\hat{f}_1(x) = -\frac{i}{2}v' = 0,$

$\hat{f}_2(x) = -\frac{1}{4}(v'' - 2v^3) + c_2^D v = 0,$
$\hat{f}_3(x) = \frac{i}{8}(v''' - 6v^2 v') - \frac{ic_2}{2} v' = 0,$

Esta jerarquía de ecuaciones es completamente integrable, y posee una formulación de par de Lax que relaciona soluciones estacionarias de la Jerarquía de mKdV con integrales de movimiento para un operador de Dirac en una dimensión cuyo potencial es transparente o finite-gap. Como veremos la interpretación de este Hamiltoniano como operador de Bogoliubov-de Gennes relaciona los superpotenciales con soluciones estacionarias del modelo de Gross-Neveu.

En este caso el par de Lax toma la siguiente forma

$$P_{n+1}^D = \sum_{\ell=0}^{n+1} \begin{pmatrix} \hat{g}_\ell(x) & \hat{f}_{\ell-1}(x) \\ \hat{f}^*_{\ell-1}(x) & \hat{g}_\ell(x) \end{pmatrix} \sigma_3 H^{D\,n+1-\ell}, \tag{5.5}$$

$$H^D = \begin{pmatrix} -i\frac{d}{dx} & v(x) \\ v(x) & i\frac{d}{dx} \end{pmatrix}, \quad H^D \Psi = E^D \Psi\,. \tag{5.6}$$

cuya ecuación de Lax en el caso estacionario es

$$[P_{n+1}^D, H^D] = \begin{pmatrix} 0 & 2\hat{f}_{n+1}(x) \\ -2\hat{f}^*_{n+1}(x) & 0 \end{pmatrix} = \mathbf{0}, \tag{5.7}$$

el operador 5.20 corresponde a un operador de Dirac con potencial escalar. Los coeficientes c_k^D están relacionados con los bordes del espectro de esteo perador en la forma

$$c_k^D = \sum_{\substack{i=j_0,j_1,\ldots,j_n=0 \\ j_0+j_1+..+j_n=k}}^{k} 2^{-2k} \prod_{i=0}^{2n+1} \frac{(2j_i)!}{(j_i!)^2(2j_i-1)} E_i^{D\,j_i}. \tag{5.8}$$

Ambos operadores en el par de Lax cumplen una relación de tipo Burchnall-Chaundy

$$P_{n+1}^{D\,2} = \prod_{\ell=0}^{2n+1} (H^D - E_\ell), \tag{5.9}$$

que define la siguiente curva hiper-éliptica

$$y^{D\,2} = \prod_{\ell=0}^{2n+1} (z^D - E_\ell^D), \tag{5.10}$$

util en el método álgebro geométrico de solución de las ecuaciones de la jerarquía de mKdV.

5.1. Transformación de Miura

La transformación de Miura es definida por la ecuación de Riccati

$$u = v^2 - v_x \tag{5.11}$$

entre las soluciones v de mKdV

$$\hat{f}_{2n+1} = 0. \tag{5.12}$$

y soluciones u de KdV

$$f_{n,x} = 0, \tag{5.13}$$

la ecuación de mKdV (5.12) es invariante bajo el cambio $v \to -v$, luego la transformación (5.14) permite definir

$$u^+ = v^2 + v_x, \tag{5.14}$$

y

$$u^- = v^2 - v_x. \tag{5.15}$$

ambas soluciones de la ecuación de KdV en función de v que es solución de mKdV. Consideremos ahora las ecuaciones (5.14) y (5.15) desde otra perspectiva. Asumamos que tenemos una solucón $u^+(x,s)$, y tratamos la ecuación (5.14) como una ecuación nolineal de Riccati que define la función v. Sí nosotros asumimos que $u^+(x)$ satisface la ecuación de KdV (5.13), entonces nosotros encontramos que la función $v(x)$ definida por (5.14) satisface no mKdV, pero la ecuación

$$f_{n,x}(u^+) = i(2v + \partial_x)\hat{f}_{2n-1} = 0. \tag{5.16}$$

Si en vez de (5.14) definimos una función v a partir de (5.15), y asumimos que $u^-(x,s)$ satisface la ecuación de KdV, entonces en vez de (5.16) nosotros obtenemos la ecuación

$$f_{n,x}(u^-) = i(2v - \partial_x)\hat{f}_{2n-1} = 0. \tag{5.17}$$

Asumamos ahora que tenemos dos funciones $u^+(x)$ y $u_-(x)$ dadas por (5.14) y (5.15) en terminos de una función $v(x)$, y supongamos que ambas funciones u^+ y u^- satisfacen la ecuación de KdV. En este caso $v(x,s)$ debe satisfacer *simultaneamente* las dos ecuaciones (5.16) y (5.17). Sumando estas dos ecuaciones obtenemos $4v\hat{f}_i = 0$, que implca que v tiene que satisfacer la ecuación de mKdV (5.12).

5.1.1. Modelo de Gross-Neveu y simetrias de la partícula de espín-$\frac{1}{2}$ relativista en una linea

Esta esupersimetría exótica con supercargas de orden uno e integrales de Lax-Novikov abre nuevas maneras de construir condensados auto-consistenter basados en la ecuación de Bogoliubov-de Gennes y asociado con ello nuevas soluciones al modelo de Gross-Neveu [30, 22, 36],[27]-[24]. Estas corresponden a soluciones de tipo kink o kink-antikink en un fondo cuaci-periódico en dependencia si la supersimetría exótica es exacta o es espontáneamente rota. El modelo de Gross-Neveu esta definido por el Lagrangiano

$$\mathcal{L}_{\rm GN} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2 , \tag{5.18}$$

corresponde a un modelo de interacción no lineal entre fermiones de distintos sabores, el cual en limite t'Hooft $N \to \infty$, $g^2 N \sim 1$ es renormalizable y permite una generación dinámica de masa. Una version efectiva de la acción asociada a dicho Lagragiano en función del condensado no lineal de fermiones $\Delta = -g^2 \left(\bar{\psi}\psi\right)$ está dada por

$$S_{\text{eff}} = -\frac{1}{2g^2} \int \Delta^2 - i\, N\, \ln \det \left[i \not{\partial} - \Delta \right] ,$$

la variación de esta acción nos da la ecuación de consistencia

$$\Delta(x) = -iNg^2 \text{tr}_{D,E} \left[\gamma^0 R(x;E) \right] , \tag{5.19}$$

$\gamma_0 = \sigma_1$ y $\gamma_2 = -i\sigma_2$. Áca, $R(x;E)$ es la resolvente diagonal $R(x;E) \equiv \langle x|(H-E)^{-1}|x\rangle$ [35] del operador Hamiltoniano de tipo Bogoliubov de Gennes,

$$H = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta(x) & i\frac{d}{dx} \end{pmatrix}, \quad H\Psi = E\Psi \, . \tag{5.20}$$

La resolvente es una matriz 2×2 que satisface las sigentes propiedades algebraicas $R = R^\dagger$, $\text{tr}_D \left(R\sigma_3\right) = 0$, $\det R = -\frac{1}{4}$ y tambien la ecuación de Dickey-Eilenberger

$$\frac{\partial}{\partial x} R\,\sigma_3 \quad = \quad i \left[\begin{pmatrix} E & -\Delta \\ \Delta & -E \end{pmatrix}, R\,\sigma_3 \right], \tag{5.21}$$

la expansión en serie de la resolvente diagonal $R = \sum_n^\infty r_n(x)/E^n$ puede ser truncada con el fin de buscar soluciones para el condensado $\Delta(x)$.

$$R_n(x;E) = \mathcal{N}(E) \sum_{l=0}^{n} E^{n-l} \begin{pmatrix} \hat{g}_l(x) & \hat{f}_{l-1}(x) \\ \hat{f}^*_{l-1}(x) & \hat{g}_l(x) \end{pmatrix} . \tag{5.22}$$

acá $\hat{g}_l(x)$ y $\hat{f}_l(x)$ están definidos recursivamente

$$\hat{f}_l = -\frac{i}{2}\hat{f}'_{l-1} + \Delta\, \hat{g}_l, \quad \hat{g}_l = i \int \left(\hat{f}_{l-1} - \Delta\, \hat{f}^*_{l-1} \right) dx + c_l$$

$$\hat{f}_{-1} = 0, \hat{f}_0 = \Delta(x), \quad \hat{g}_0 = c_0 = 1. \tag{5.23}$$

Si se exige que $\Delta(x)$ sea una solución de la jerarquía inhomogénea y estacionaria de mKdV la ecuació n de consistencia 5.19 se convierte en un sistema de ecuaciones que define la ocupación de las bandas y estados ligados por los fermiones de distinto sabor.

El conjunto de soluciones de la jerarquía estacionaria de mKdV está dado por los superpotenciales de las transformaciones de Darboux entre potenciales multisolitónicos transparentes o en fondo finite-gap del operador Hamiltoniano de Schrödinger estudiados en los capítulos anteriores, debemos diferenciar entre dos posibles casos i) transformaciones auto Darboux , construidas usando un estado de la banda prohibida más baja de $H_{g,l}$ sin ceros y energía $\breve{z}_1$, el cual solo crece o decrece exponencialmente en x y ii) cuando sumamos un solitón usando un estado sin

ceros que es la mezcla de un estado creciente con un estado decreciente de la banda prohibida más baja de $H_{g,l-1}$ con energı'a z_l. estos casos corresponden a pares de transformaciones de Miura $u_{g,l} - z_l = v^{K2} - v_x^K$ y $u_{g,l-1} - z_l = v^{K2} + v_x^K$; y $u_{g,l} - \breve{z}_1 = v^{K-AK2} - v_x^{K-AK}$ y $\tilde{u}_{g,l}^{\breve{r}_1} - \breve{z}_1 = v^{K-AK2} + v_x^{K-AK}$ respectivamente. Para el Hamiltoniano extendido una redefinición de $\mathcal{P}_1$ y $\breve{\mathcal{P}}_1$ juega el rol de carga central y por lo tanto de integral de movimiento para los operadores Hamiltonian de tipo Dirac $\mathcal{Q}_a$ and $\breve{\mathcal{Q}}_a$ respectivamente, además la naturaleza de tipo kink del superpotencial en $\mathcal{Q}_a$ genera una reducción de orden espontánea del operador de Lax-Novikov de KdV extendido debido a que en este caso es posible escribir la factorización $\mathcal{P}_1 \propto \mathcal{Q}_1\mathcal{S}_1$ que desde el punto de vista del operador de Dirac como Hamiltoniano permite la reducción $\mathcal{P}_1 \to \mathcal{S}_1$. En el caso en que el superpotencial o potencial de Dirac sea de forma kink-antikink la proporcionalidad anterior no es valida siendo imposible la reducción de orden.

En el primer caso el Hamiltoniano extendido $\breve{\mathcal{H}} = \mathrm{diag}(H_{g,l}, \tilde{H}_{g,l}^{\breve{r}_1})$ define una integral de Lax-Novikov en la forma

$$\breve{\mathcal{P}}_1 = \begin{pmatrix} P_{2g+2l+1} & 0 \\ 0 & \tilde{P}_{2g+2l+1}^{\breve{r}_1} \end{pmatrix}, \tag{5.24}$$

cuya condición 5.16 y 5.17 toma la forma

$$[\mathcal{P}_1, \breve{\mathcal{H}} - \breve{z}_1] = [\mathcal{P}_1, \mathcal{Q}_1] = \mathbf{0}, \tag{5.25}$$

lo que corresponde a la formulación de par de Lax de ambas jerarquías KdV y mKdV

$$\breve{\mathcal{Q}}_1 = \begin{pmatrix} 0 & X_{1,l}(\breve{r}_1)^{\#} \\ X_{1,l}(\breve{r}_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{d}{dx} + v_{g,l}^{K-AK} \\ \frac{d}{dx} + v_{g,l}^{K-AK} & 0 \end{pmatrix}, \quad \breve{\mathcal{Q}}_1 = e^{i\frac{\pi}{4}\sigma_1} H_{K-AK}^D e^{-i\frac{\pi}{4}\sigma_1}, \tag{5.26}$$

$$H_{K-AK}^D = \begin{pmatrix} -i\frac{d}{dx} & v^{K-AK} \\ v^{K-AK} & i\frac{d}{dx} \end{pmatrix}, \tag{5.27}$$

acá

$$v = \pm v^{K-AK} = \mp \frac{d}{dx} \ln(\mathbb{A}_l \psi(\breve{r}_1)), \tag{5.28}$$

son soluciones de $\hat{f}_{2g+2l+1}(v) = 0$ cuyos coeficientes c_k^D están dados por las energías $\partial\sigma(\breve{\mathcal{Q}}_1)$,

$$\begin{aligned} \sigma(\breve{\mathcal{Q}}_1) \quad &= \quad (-\infty, -\sqrt{E_0 - \breve{z}_1}] \cup [-\sqrt{E_1 - \breve{z}_1}, -\sqrt{E_2 - \breve{z}_1}] \cup \dots \\ &\cup \quad [-\sqrt{E_{2n-1} - \breve{z}_1}, -\sqrt{E_{2n} - \breve{z}_1}] \cup [\sqrt{E_{2n} - \breve{z}_1}, \sqrt{E_{2n-1} - \breve{z}_1}] \cup \dots \\ &\cup \quad [-\sqrt{E_2 - \breve{z}_1}, -\sqrt{E_1 - \breve{z}_1}] \cup [\sqrt{E_0 - \breve{z}_1}, \infty) \\ &\cup_{j=1}^{l} \quad \{-\sqrt{z_j - \breve{z}_1}, \sqrt{z_j - \breve{z}_1}\}, \end{aligned} \tag{5.29}$$

acá $\{E_0, \dots, E_{2n}\} = \partial\sigma(H_{g,0})$ y $z_j = z(r_{j,1}) = z(r_{j,2})$ son las energías del borde del espectro de $H_{g,l}$ note que para una base de autoestados común las energías $\breve{E}^D$ de $\breve{\mathcal{Q}}_1$ y las energías $\breve{E}$ de $\breve{\mathcal{H}}$ están relacionadas en la forma $\breve{E}^{D2} = \breve{E} - \breve{z}_1$.

Por otro lado en el caso ii) el Hamiltoniano extendido toma la forma $\mathcal{H} = \mathrm{diag}(H_{g,l}, H_{g,l-1})$ y su integral de Lax-Novikov puede ser escrita en la forma

$$\mathcal{P}_1 = \begin{pmatrix} P_{2g+2l+1} & 0 \\ 0 & (H_{g,l-1} - z_l)P_{2g+2l-1} \end{pmatrix}, \quad \mathcal{P}_2 = \sigma_3 \mathcal{P}_1. \tag{5.30}$$

En la definición de $\mathcal{P}$ introducimos el término $(H_{g,l-1} - z_l)$, esto permite tener operadores del mismo orden en los elementos diagonales lo cual sera necesario en el próximo análisis

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & A_l \\ A_l^{\#} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{dx} + v_{g,l}^K \\ -\frac{d}{dx} + v_{g,l}^K & 0 \end{pmatrix}, \quad \mathcal{Q}_1 = e^{-i\frac{\pi}{4}\sigma_1} H_K^D e^{i\frac{\pi}{4}\sigma_1}, \tag{5.31}$$

$$\mathcal{S}_1 = \begin{pmatrix} 0 & A_l P_{2g+2l-1} \\ P_{2g+2l-1} A_l^{\#} & 0 \end{pmatrix}, \quad \mathcal{S}_2 = i\sigma_3 \mathcal{S}_1, \tag{5.32}$$

$$[\mathcal{P}_1, \mathcal{H} - z_l] = 0, \quad [\mathcal{P}_1, \mathcal{Q}_a] = 0, \tag{5.33}$$

para v^K es facil mostrar que $\mathcal{P}_1 = \mathcal{Q}_1 \mathcal{S}_2$ y entonces tomando $\mathcal{Q}_1$ como Hamiltoniano de Dirac su integral de movimiento irreducible de tipo mKdV Lax-Novikov no es $\mathcal{P}_1$ sino que $\mathcal{S}_1$,

$$[\mathcal{Q}_1, \mathcal{S}_1] = 0. \tag{5.34}$$

Acá

$$v = \pm v^K = \mp \frac{d}{dx} \ln(\mathbb{A}_{l-1} \psi_{a_{l,1},a_{l,2}}(r_{l,1}, r_{l,2}, x)), \tag{5.35}$$

son soluciones de $\hat{f}_{2g+2l}(v) = 0$ con coeficientes c_k^D dados por los bordes del espectro de $\partial\sigma(\mathcal{Q}_1)$,

$$\begin{aligned} \sigma(\mathcal{Q}_1) \quad &= \quad (-\infty, -\sqrt{E_0 - z_l}] \cup [-\sqrt{E_1 - z_l}, -\sqrt{E_2 - z_l}] \cup \dots \\ &\cup \quad [-\sqrt{E_{2n-1} - z_l}, -\sqrt{E_{2n} - z_l}] \cup 0 \cup [\sqrt{E_{2n} - z_l}, \sqrt{E_{2n-1} - z_l}] \cup \dots \\ &\cup \quad [-\sqrt{E_2 - z_l}, -\sqrt{E_1 - z_l}] \cup [\sqrt{E_0 - z_l}, \infty) \\ &\cup_{j=1}^{l-1} \quad \{-\sqrt{z_j - z_l}, \sqrt{z_j - z_l}\}. \end{aligned} \tag{5.36}$$

La existencia de un estado con energía cero está directamente relacionado con la naturaleza de kink o antikink de $\pm v^K$, para una base común de estados las energías E^D de $\mathcal{Q}_1$ y las energías E de $\mathcal{H}$ están relacionadas por $E^{D2} = E - z_l$.

La soluciones v de la jerarquía inhomogénea y stationary de mKdV tiene la propiedad caracteristica de que su espectro es siempre simétrico con respercto a $E^D = 0$. Esto particularmente nos permite relacionar estos sistemas de Dirac multi-kink-antikink en fondo finite-gap y caracterizarlos por su carga topológica pudiendo ser cero o uno dependiendo si el fondo es de tipo kink o kink-antikink. La existencia de esta integral no trivial para operadores de tipo Dirac permite una construcción de supersimetría no lineal extendida para pares de operadores Hamiltonianos de Dirac multi-kink-antikink en fondo finite-gap crum darboux conectados. Los resultados obtenidos

aqui pueden ser interesantes, particularmente, desde la perspectiva de sus aplicaciones de física de nanoestructuras de carbono.

En muchas aplicaciones físicas los potenciales transparentes y finite-gaps $v(x)$ aparecen como soluciones estacionarias para condensados auto-consistentes de fermiones. Estas estan dadas por el sistema de ecuaciones de Dirac en (1+1)D

$$(i\partial\!\!\!/ - v)\,\psi_\alpha = 0, \tag{5.37}$$

sujeto a las condiciones

$$v = -g^2 \sum_{\alpha=1}^{N} \sum_{\mathrm{occ}} \bar{\psi}_\alpha \psi_\alpha \,. \tag{5.38}$$

Acá $\sum_{\alpha=1}^{N}$ corresponde a la suma en la degeneración de estados, con α denotando una generalización de sabores (posiblemente, incuyendo el espín), y $\sum_{\mathrm{occ}}$ es una suma sobre los niveles de energía ocupados por cada sabor[1]. Particularmente, esta ecuación aparece en superconductividad, en física de conducción en polimeros, y el modelo de Gross-Neveu que estudiamos en particular acá.

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[1] Note la similitud entre (5.37) y (5.38) con la ecuación (1.39) y (1.32).

Conclusiones

A lo largo de esta tesis hemos mostrado diversas manifestaciones de la transformación de Darboux como el eje central de la existencia de simetrías en mecánica cuántica en una dimensión. Mediante el mecanismo de cadenas de Darboux se estudian las integrales de Lax-Novikov para los potenciales completamente transparentes y finite-gap [41, 7]. Por otro lado la transformación de Darboux permite la construcción de mecánica cuántica supersimétrica y la contrucción recursiva de familias infinitas de sistemas exactamente solubles a partir de un sistema exactamente soluble inicial. En esta tesis son observadas supersimetrías exóticas entre sistemas exactamente solubles correspondiente a defectos solitónicos de la particula libre, el potencial de Lamé o los potenciales en la forma de Its-Matveev.

Su generalización en forma de la transformación de Crum-Darboux permite una realización no lineal de supersimetría, lo cual es un objeto poco estudiado y talvez pueda cumplir un rol fundamental en teorías modernas, ya los coeficientes de estructura que conocemos de las álgebras de Lie en este caso pasan a ser objetos dependientes de los generadores de la simetría. Los caminos necesarios a seguir para estudiar todos los aspectos de estas supersimetrías exóticas vienen de la mano con generalizaciones de la transformación de Darboux para operadores Hamiltonianos en más dimensiones, siendo relevante, por ejemplo: una realización nolineal de alguna super extención del grupo de Poincare en $n+1$ dimensiones, además de al menos dejar la duda de como podría afectar esta no linealidad en teoría cuántica de campos.

Lo interesante de este método es que las transforaciŕes de Darboux mediante el vestimiento de Darboux preservan las simetrías pero en general deforman los coeficientes de estructura en forma nolineal (supersimetría). Estos sistemas son dependientes de los datos de dispersión de los estados utilizados en las transformaciones de Darboux, siendo las simetrías de cada sistema en particular suceptible a cambios considerables ante deformaciones en los datos espectrales [5]. Pudiendo suceder reducciones espontáneas de orden de no linealidad en los coeficientes de estructura además de rompimiento espontaneo de simetría [8].

Es importante notar que las integrales de Lax-Novikov a nivel clásico no corresponden más que a potencias del momento lineal dado que en este limite $\hbar \to 0$ lo que hace desaparecer a todos estos potenciales reconstruyendo el Hamiltoniano de la partícula libre clásica en todos estos casos. Es necesario un estudio más a fondo, pero cada uno de estos potenciales es una

representación no lineal del álgebra de Lie asociada al grupo de simetrías de la partícula libre

$$[H, P] = 0 \tag{5.39}$$

La relación íntima de estas simetrías con los sistemas integrables de KdV y mKdV conectan los superpotenciales no singulares de la supersimetría exótica acá mostrada, con diversos problemas exactamente solubles en diversas áreas de la física, siendo posible relacionar algunos de estos potenciales y superpotenciales con solitones en agua poco profunda, fraccionalización de carga, modelos de conducción en poliacetileno, soluciones analíticas de conducción en grafeno, la propagación de la luz en fibra óptica y soluciones estacionarias del modelo de Gross-Neveu, entre otros.

Se ha generalizado el proceso de construcción de defectos solitónicos en fondos finite-gap tanto para operadores de Dirac con potencial escalar como para operadores de Schrödinger [4]. En el caso de la ecuación de KdV dependiente del tiempo se ha estudiado la propagación de dichos defectos solitónicos. La propagación de solitones en el caso dela ecuación de mKdV ha sido resuelto en detalles en el marco de este proyecto de tesis y puede ser encontrado en [9].

Se han construido potenciales transparentes y en fondo finite-gap para sistemas de fermiónicos los cuales presentan una integral de movimiento correspondiente al par de Lax de la jerarquía de ecuaciones de mKdV [6]. La presencia de esta integral permite la extensión de la supersimetría exótica $N = 4$ para Hamiltonianos extendidos por dos copias de operadores de BdG o operadores Hamiltonianos de Dirac en 1+1D.

El método supersimétrico ha mostrado ser útil para resolver problemas de interacción nolineal entre bosones, y problemas de interacción no lineal entre fermiones. En esta dirección es interesante buscar aplicaciones del métodos supersimétrico a problemas de interacción no lineal entre bosones y fermiones.

Existe una generalización de la transformación de Darboux para operadores de Dirac en 1+1D. A partir de Dicha generalización se pueden construir potenciales transparentes más generales que los presentados aquí, estos potenciales no son soluciones de la jerarquía de mKdV, sino que de la jerarquía de ecuaciones de ZS-AKNS [54, 42], En general estos potenciales transparentes corresponden a potenciales pseudo escalares, y para ellos también es posible constuir una supersimetría extendida $N = 4$ debido a la existencia de una formulación de par de Lax dicha jerarquía de ecuaciones integrables. Los potenciales estacionarios asociados esta jerarquía corresponderán a soluciones estacionarias del modelo de Gross-Neveu quiral o Nambu-jona-Lasinio en 1+1D.

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